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ON THE DISTRIBUTION OF ENERGY IN NOISE-AND SIGNAL-MODULATED WAVES I. AMPLITUDE MODULATION*

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1. Introduction. Because noise is an inevitable and undesirable companion of intelligence transmitted or received by electronic systems, it is essential for any proper theory of communication to provide suitable methods for studying the physical properties of a noise wave and its interaction with a desired signal. On the one hand a successful technique of measurement is required to control or minimize the noise, and on the other an adequate theory is necessary to guide experiment and interpret the data. Accordingly, the purpose of the present paper is to present a number of new results, obtained by the analytical methods developed in recent years, [2-15, 17-20] for the following important problems. (In all cases considered here the noise is assumed to belong to the fluctuation type characteristic of shot and thermal noise, which are described by a normal random process [5, 10]. Impulsive noise, such as atmospheric and solar static, is not treated, although the general methods of analysis remain the same.) Our interest here is confined mainly to amplitude-modulated waves, specifically,

(i) *carrier amplitude-modulated by noise*: this problem considers the amplitude distortion by noise of the carrier wave as the mechanism producing the modulation, and the important case of over-modulation is also examined. This example is of particular interest when normal random noise is used as an approximate model of speech,† or as a form of interference.

(ii) *carrier amplitude-modulated by a signal and noise*: this is a frequent case in practice where a certain amount of noise accompanies the desired signal in the process of modulation. Included also is the problem of speech (normal random noise) accom-

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†Recent experiments of W. B. Davenport, Jr. ("A study of speech probability distributions," Technical Report 148, Research Laboratory of Electronics, (MIT) August 25, 1950) indicate that speech is more satisfactorily described statistically in terms of an impulsive or "static" noise model, where overlapping among individual (and independent) pulses is assumed to be small, of the order of 30-50 per cent of the time. This is different from the usual model of fluctuation or normal random noise, which assumes complete and highly multiple overlapping between the elementary transients. However, because normal random noise has the great advantage of mathematical simplicity, its use as a speech model seems justified on this and physical grounds, at least as a first approximation.

panied by noise; the two noise waves are assumed to be uncorrelated. In a later paper* is considered

(iii) *simultaneous amplitude- and angle-modulation of a carrier by noise*: here the modulating noise waves are correlated, and there may be in general a phase lag of one modulation with respect to the other. The results are of particular interest in connection with the problem of the noise in magnetron generators, in which a simultaneous amplitude- and angle- (i.e., phase- or frequency-) modulation of the oscillations due to the inherent or primary noise of the tube is known to occur.

[A discussion of the problem of carriers angle-modulated by signal and noise has been given elsewhere in a recent report (D. Middleton, 1)].

We remark further that, apart from the specific applications to problems (i)-(iii), the results are needed in the general theory of noise measurement, for here the central objective is to be able to determine by measurements on the output wave, following various linear and nonlinear operations (such as amplification, rectification, clipping, mixing, modulation, discrimination, etc.), the "structure" of the original input disturbance, i.e., whether or not it is an amplitude- or frequency-modulated wave, how the noise and signal occur together, and other qualitative and quantitative data.

The quantities of chief physical interest are (a), the *mean* or steady component of the disturbance, (b), the *mean intensity* of the wave, and (c), the *spectral distribution* $W(f)$ of the mean intensity. This latter quantity is in fact sufficient to give us the other two; the mean intensity is obtained as the area under the spectral density curve $W(f)$, while the (square of the) steady component is given by the constant (or frequency-independent) term in the expression for the spectrum. It is assumed that we are dealing with a stationary (and ergodic) random process, namely, a process for which the underlying mechanism does not change with time. Then time averages and ensemble or statistical averages are equivalent, [20] to within a set of random functions of probability zero, so that if we represent our stochastic, time-dependent disturbance by $y(t)$ we may write the steady component (a) as**

$$\langle y(t_0)\rangle_{av.} \equiv \lim_{T\to\infty} T^{-1}\int_0^T y(t_0)\,dt_0 = \langle y(t_0)\rangle_{s.av.} \equiv \int_{-\infty}^{\infty} yW_1(y)\,dy, \tag{1.1}$$

and the mean intensity (b)

$$\langle y(t_0)^2\rangle_{av.} \equiv \lim_{T\to\infty} T^{-1}\int_0^T y(t_0)^2\,dt_0 = \langle y(t_0)^2\rangle_{s.av.} \equiv \int_{-\infty}^{\infty} y^2W_1(y)\,dy; \tag{1.2}$$

$W_1(y)\,dy$ is the probability that (at any initial time t_0) y lies in the range y, $y + dy$. The moment of greatest interest, however is given by

$$R(t) \equiv \langle y(t_0)y(t_0 + t)\rangle_{av.} \equiv \lim_{T\to\infty} T^{-1}\int_0^T y(t_0)y(t_1 + t)\,dt_0\,. \tag{1.3}$$

The quantity $R(t)$ is the *auto-correlation function* of y and may be found statistically when the second-order probability density $W_2(y_1\,,\,y_2\,;\,t)$ is known; W_2 has the following interpretation:

*We distinguish here between time and statistical averages by $\langle\ \rangle_{av.}$ and $\langle\ \rangle_{s.av.}$ respectively.

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The equivalence of these averages follows from the ergodic theorem [2, 20].

$W_2(y_1, y_2; t)\, dy_1\, dy_2 =$ the joint probability that at some (initial) time t_0, $y(= y_1)$ lies in the range $(y_1, y_1 + dy_1)$ and at a later time $t_0 + t$, $y(=y_2)$ falls in the interval $(y_2, y_2 + dy_2)$. (1.4)

Because time and ensemble averages are equivalent here, Eq. (1.3) becomes

$$R(t) = \langle y_1 y_2 \rangle_{\text{e.av.}} = \iint_{-\infty}^{\infty} y_1 y_2 W_2(y_1, y_2; t)\, dy_1\, dy_2, \tag{1.5}$$

and since the process is stationary, the initial times t_0 do not enter: one is concerned only with the time intervals (t) between observations.

Knowledge of the correlation function $R(t)$ is important, for by the theorem of Wiener [14] and Khintchine [15] the mean intensity spectrum follows at once as the cosine Fourier transform of $R(t)$, namely

$$W(f) = 4 \int_0^{\infty} R(t) \cos \omega t\, dt, \qquad (\omega = 2\pi f) \tag{1.6a}$$

with the inverse relation

$$R(t) = \int_0^{\infty} W(f) \cos \omega t\, df. \tag{1.6b}$$

To determine the desired energy spectrum $W(f)$ the simplest procedure is first to obtain the correlation function and then apply (1.6a). Note from (1.6b) that setting $t = 0$ in $R(t)$ gives the mean total intensity of the random wave, namely

$$R(0) = \int_0^{\infty} W(f)\, df = \lim_{t \to 0} \iint_{-\infty}^{\infty} y_1 y_2 W_2(y_1, y_2; t)\, dy_1\, dy_2 = \langle y^2 \rangle_{\text{e.av.}} \tag{1.7}$$

which is the area under the spectral distribution curve $W(f)$, as expected. On the other hand, allowing t to become infinite in $R(t)$ yields the steady component $\langle y \rangle_{\text{e.av.}}$, since $\lim_{t \to \infty} W_2(y_1, y_2; t) = W_1(y_1) W_1(y_2)$, so that (1.5) becomes

$$\lim_{t \to \infty} R(t) = \iint_{-\infty}^{\infty} y_1 y_2 W_1(y_1) W_1(y_2)\, dy_1\, dy_2 = \langle y \rangle_{\text{e.av.}}^2, \tag{1.8}$$

from (1.1). For a pure noise wave $\langle y \rangle_{\text{e.av.}}$ vanishes, as there are no steady components. However, when y does not represent a purely stochastic variable, but contains steady and periodic terms as well, $\lim_{t \to \infty} R(t)$ will not die down in time, but will oscillate indefinitely. If $R(t)$ is then expanded in a Fourier series, the coefficient of each periodic component represents the mean power (or intensity, as the case may be)* associated with that component; setting $t = 0$ in $R(t)$ still gives the mean total power in the wave.

In a similar way we may find the correlation function for a general function $g(y)$ of the random variable y. By definition (3, 10) we have

$$\begin{aligned} R(t) &\equiv \langle g[y(t_0)] g[y(t_0 + t)] \rangle_{\text{av.}} = \langle g(y_1) g(y_2) \rangle_{\text{e.av.}} \\ &= \iint_{-\infty}^{\infty} g(y_1) g(y_2) W_2(y_1, y_2; t)\, dy_1\, dy_2. \end{aligned} \tag{1.9}$$

*The mean intensity may be expressed in units of power or mean square amplitude, appropriate to the problem in question.

The spectrum follows from (1.6a). For problems (i)-(iii) the modulated wave $V(t)$ is expressed as a function of a statistical variable y, and the choice of $g(y)$ is based on the pertinent physical model which describes the problem. In general, $g(y)$ is not a linear function of y, and so the evaluation of the auto-correlation function becomes difficult. These remarks are illustrated in the following sections.

The main results of the analysis of a carrier wave amplitude-modulated by normal random noise, or a signal and noise, show that the amplitude-distortion characteristic of over-modulation spreads the spectrum but not significantly; the additional noise components are due to $(n \times n)$ noise modulation products. Furthermore, when a modulating signal accompanies the noise, distortion of the signal also occurs, and $(s \times n)$ as well as $(n \times n)$ noise harmonics are produced. Expressions for the mean total power, the mean continuum power, the mean carrier power, and the mean power in the discrete portions of the spectrum are given, along with a detailed treatment of the spectral distribution of the wave's energy. In the following sections, a discussion of the limiting cases of weak noise, strong signal, etc., is included, and a number of figures illustrate the principal results.

2. Carrier amplitude-modulated by noise. We represent the *IF* (or *RF*) wave by a complex disturbance

$$g(y) = V(t) = A_0(t) \exp (i\omega_0 t), \qquad (\omega_0 = 2\pi f_0), \tag{2.1}$$

where $A_0(t)$ is a real quantity. The amplitude modulation is specifically

$$\left.\begin{aligned} A_0(t) &= A_0(1 + kV_N(t)), \qquad & y = kV_N(t) \geq -1 \\ &= 0, & y = kV_N(t) \leq -1 \end{aligned}\right\}, \tag{2.2}$$

in which $V_N(t)$ is a normal random noise voltage (or current) and k is a modulation index, with dimensions $(\text{volts})^{-1}$. When the instantaneous amplitude $V_N(t)$ is less than $-1/k$, *over-modulation* occurs, and the signal generator does not oscillate until $V_N(t)$ is once more greater than $-1/k$. Since we assume a purely normal random noise, large and even infinite amplitudes are possible, and consequently we may expect over-modulation for a noticeable part of the time, unless the modulating noise is weak. The analysis of this and succeeding sections assumes the common type of modulation in which the instantaneous amplitude of an oscillator's output is modified according to some signal or other low-frequency disturbance applied to a suitable control grid. Frequently, however, a modulated output is produced by applying the *sum* of the separately generated oscillations and the modulation to the input of a (half-wave linear) rectifier. The tube acts now as a mixing device, which yields a suitably modulated output carrier wave *only if the original carrier oscillations are very intense relative to the modulation.* Otherwise one obtains serious distortion due to the significant additional harmonics generated in the nonlinear mixing of the signal (noise), carrier, and background noise. Thus, if a mixer is used, Eqs. (2.1) and (2.2) apply here approximately, provided the modulation is weak, while (2.1) and (2.2) are valid models for all degrees of carrier and modulation strengths when the alternative system of a modulated oscillator is employed.

Let us consider first the simpler and less general situation in which the *r-m-s* noise amplitude $\langle V_N(t)^2\rangle_{s.av}^{1/2}$ is very much less than $1/k\ 2^{1/2}$; this means that for an overwhelmingly large percentage of the time the instantaneous amplitude $kV_N(t)$ is less than unity

and therefore over-modulation is for all practical purposes ignorable. The exact expression (2.1) then becomes simply

$$A_0(t) = A_0(1 + kV_N(t)) = A_0(1 + y), \qquad (-\infty < y = kV_N(t) < \infty), \tag{2.3}$$

provided $(0 \leq k(2\psi)^{1/2} \ll 1)$, where by ψ we abbreviate $\langle V_N(t)^2 \rangle_{\text{s.av.}}$. To see how strict the condition $(0 \leq k(2\psi)^{1/2} \ll 1)$ must be, we ask what fraction of the time α $(0 \leq \alpha \leq 0.5)$, $V(t)$ exceeds an amplitude $V_0 = 1/k$. Since $V(t)$ is normally distributed, we have at once

$$\alpha = \frac{1}{(2\pi\psi)^{1/2}} \int_{V_0 = 1/k}^{\infty} \exp(-V^2/2\psi)\, dV = \frac{1}{2}[1 - \Theta(1/k(2\psi)^{1/2})], \qquad (\psi \equiv \langle V_N(t)^2 \rangle_{\text{s.av.}}), \tag{2.4}$$

where Θ is the familiar error function

$$\Theta(x) = 2\pi^{-1/2} \int_0^x \exp(-z^2)\, dz, \qquad \text{and} \qquad \phi^{(n)}(x) \equiv \frac{d^n}{dx^n} \frac{\exp(-x^2/2)}{(2\pi)^{1/2}}. \tag{2.4a}$$

With the help of tables of Θ we readily determine $k(2\psi)^{1/2}$ corresponding to a chosen value of α. We see for example that when $k(2\psi)^{1/2} \leq 0.6$, over-modulation occurs less than 1 per cent of the time, and so for most purposes Eq. (2.3) may replace the more general relation (2.2) when $\langle V_N(t)^2 \rangle_{\text{s.av.}} \leq 0.18/k$.

The autocorrelation function of the modulated wave is therefore from (1.3) and (2.3)

$$\begin{aligned} R(t) &= (A_0^2/2)\, Re\{\exp(-i\omega_0 t)[1 + k^2 \langle V_N(t_0) V_N(t_0 + t) \rangle_{\text{av.}}]\} \\ &= (A_0^2/2) \cos \omega_0 t \{1 + k^2 R_0(t)_N\}, \end{aligned} \tag{2.5}$$

in which $R_0(t)_N = \langle V_N(t_0) V_N(t_0 + t) \rangle_{\text{s.av.}}$ is the auto-correlation function of the modulating noise wave. The mean intensity spectrum is from (1.6)

$$W(f) = (A_0^2/2)\, \delta(f - f_0) + A_0^2 k^2 \int_0^\infty R_0(t)_N \cos(\omega_0 - \omega)t\, dt, \tag{2.6}$$

where we neglect the contribution from $\cos(\omega_0 + \omega)t$ in (2.6), since the spectral width $(\backsim\omega_b)$ of the noise is much less than the carrier frequency $(= f_0)$. The carrier power is unchanged, viz., $W_{f_0} = A_0^2/2$, while the amount of power W_c in the continuum is distributed symmetrically about f_0, with a total intensity $A_0^2 k^2 \psi/2$. [Unlike frequency- or phase-modulation (cf. secs. 2, 3 of ref. [1]) there is a larger amount of energy in the wave after modulation than before, and all of the additional power appears in the continuous spectrum.] Now $R_0(t)_N \cos \omega_0 t$ in (2.6) represents the correlation function of the continuous part of the output spectrum, which is the same as the correlation function that one obtains for a narrow-band of noise centered about f_0, and consequently yields the same *intensity* spectrum. There is, however, an important difference between the two. In the former the component at a frequency $f_0 + f'$ is correlated with the component at the image frequency $f_0 - f'$, while in the latter there is no such coherence between pairs of harmonics symmetrically located about f_0. Nevertheless, identical forms of correlation function and spectrum occur in either instance because, by definition, these quantities are squares of a modulus, from which all phase factors are excluded. On the other hand, a Fourier analysis of the noise-modulated carrier (2.1), (2.3) and the equivalent (in power) narrow-band noise centered about f_0 show at once coherence in the former and none in

the latter. This can be observed directly on a cathode-ray oscilloscope: for a noise-modulated carrier with ignorable over-modulation, the instantaneous envelope will vary randomly but there will be no change in the phase of the carrier, while in the case of the noise band the phase of the carrier will change (relatively slowly) in a random way.

We return now to the general case (2.2), which includes *over-modulation*. To represent the discontinuities in $A_0(t)$, cf. (2.2), we express $A_0(t)$ in terms of its Fourier transform

$$A_0(t) = -A_0(2\pi)^{-1} \int_{\mathbf{C}} z^{-2}\, dz \exp\left(iz[1\ kV_N(t)]\right), \tag{2.7}$$

where $\mathbf{C}$ is a contour extending along the real axis from $-\infty$ to $+\infty$ and is indented downward in an infinitesimal semicircle about the singularity at $z = 0$. The correlation function is now

$$\begin{aligned} R(t) = (A_0^2/2)\, Re\Big\{ &\exp(-i\omega_0 t)(4\pi^2)^{-1} \int_{\mathbf{C}} z^{-2}\, dz \exp\left(iz[1 + kV_N(t_0)]\right) \\ &\cdot \int_{\mathbf{C}^*} \xi^{-2}\, d\xi \exp\left(-i\xi[1 + kV_N(t_0 + t)]\right)\Big\}_{\text{stat.av.}} \end{aligned} \tag{2.8}$$

since no coherence between carrier and modulation is assumed. Here $\mathbf{C}^*$ is the contour conjugate to $\mathbf{C}$, extending from $+\infty$ to $-\infty$ and is indented *upward* in an infinitesimal semicircle about the point $\xi = 0$. Furthermore, inasmuch as $V_N(t)$ is real, $A_0(t)$ is also and so $A_0(t)^* = A_0(t)$ and we can replace the integral over $\mathbf{C}^*$ by one in $\mathbf{C}$, setting $i = -i$. The ensemble average in (2.8) may be effected in a straightforward way if we note that since $V_N(t_0)$ $(=V_1)$ and $V_N(t_0 + t)$ $(=V_2)$ are normal random variables, their joint distribution is given by a relation of the form

$$W_2(V_1\,, V_2\,; t) = [2\pi\psi(1 - r_0^2)^{1/2}]^{-1} \exp\{-[V_1^2 + V_2^2 - 2V_1V_2r_0]/2(1 - r_0^2)\psi\}, \tag{2.9}$$

and $r_0 = \langle V_1V_2\rangle_{\text{s.av.}}/\langle V^2\rangle_{\text{s.av.}} = \psi(t)/\psi$ is the normalized auto-correlation function of the modulating noise, whose mean intensity spectrum is $W(f)_N$. Substituting (2.9) into (2.8) and observing that the statistical average yields the characteristic function (cf. Eq. (2.16) of ref. [10]) for the noise $F_2(z, \xi; t)_N = \exp\{-\frac{1}{2}k^2\psi(0)(z^2 + \xi^2) - z\xi\psi(t)\}$, we have finally

$$\begin{aligned} R(t) &= (A_0^2/2)\, Re\Big\{\exp(-i\omega_0 t)(4\pi^2)^{-1} \int_{\mathbf{C}} z^{-2}\, dz \exp(iz - k^2\psi z^2/2) \\ &\qquad \cdot \int_{\mathbf{C}} \xi^{-2}\, d\xi \exp(i\xi - k^2\psi\xi^2/2 - k^2\psi(t)z\xi\Big\} \\ &= (A_0^2h_{0,0}^2/2)\cos\omega_0 t + (A_0^2/2)\sum_{n=1}^{\infty} \frac{[-k^2\psi r_0(t)]^n}{n!}\, h_{0,n}^2 \cos\omega_0 t, \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} h_{0,n} = (2\pi)^{-1} \int_{\mathbf{C}} z^{n-2} \exp(iz - k^2\psi z^2/2)\, dz &= -i^{-n}(k^2\psi)^{(1-n)/2}2^{(n-3)/2} \\ &\cdot \Big\{{}_1F_1\Big(\frac{n-1}{2}; \frac{1}{2}; -\frac{1}{2k^2\psi}\Big)\Big/\Gamma\Big(\frac{3-n}{2}\Big) \\ &+ \frac{2}{(2k^2\psi)^{1/2}}\, {}_1F_1\Big(\frac{n}{2}; \frac{3}{2}; -\frac{1}{2k^2\psi}\Big)\Big/\Gamma\Big(\frac{2-n}{2}\Big)\Big\}, \end{aligned} \tag{2.11}$$

from Eq. (A3.17) of reference (10); ${}_1F_1$ is a confluent hypergeometric function. Specifically (cf. (A.9) of ref. 10), we have for the amplitude functions $h_{0,n}$,

$$h_{0,0} = [1 + \Theta([2k^2\psi]^{-1/2})]/2 + (k^2\psi/2\pi)^{1/2} \exp(-1/2k^2\psi) \tag{2.12a}$$

$$h_{0,1} = i[1 + \Theta([2k^2\psi]^{-1/2})]/2, \tag{2.12b}$$

$$h_{0,n} = (-1)^{n/2}(k^2\psi)^{(1-n)/2}\phi^{(n-2)}([k^2\psi]^{-1/2}), \qquad (n = 2, 3, 4, 5, \cdots) \tag{2.12c}$$

(the definitions of Θ and $\phi^{(n)}$ are given in Eq. (2.4a); see also Appendix III of ref. [10]). By Eq. (1.6a) the mean intensity spectrum is the Fourier transform of (2.10), namely,

$$\begin{aligned} W(f) = (A_0^2h_{0,0}^2/2)\ \delta(f - f_0) &+ A_0^2k^2(-h_{0,1}^2)\psi \int_0^\infty r_0(t) \cos(\omega_0 - \omega)t\, dt \\ &+ A_0^2k^2\psi \sum_{n=2}^\infty \frac{\phi^{(n-2)}([k^2\psi]^{-1/2})^2}{n!} \int_0^\infty r_0(t)^n \cos(\omega_0 - \omega)t\, dt. \end{aligned} \tag{2.13}$$

When the modulating noise has a gaussian spectrum, $W_N(f) = W_0 \exp(-\omega^2/\omega_b^2)$, this becomes explicitly

$$\begin{aligned} W(f) = (A_0^2h_{0,0}^2/2)\ \delta(f - f_0) + \pi^{1/2}A_0^2k^2\psi\omega_b^{-1}\Big\{&h_{0,1}^2 \exp[-(\omega_0 - \omega)^2/\omega_b^2] \\ &+ \sum_{n=2}^\infty \frac{\phi^{(n-2)}([k^2\psi]^{-1/2})^2}{n!n^{1/2}} \exp[-(\omega_0 - \omega)^2/n\omega_b^2]\Big\}. \end{aligned} \tag{2.14}$$

Figure (2.1) shows typical intensity spectra for a number of values of $(2k^2\psi)^{1/2}$ between 0 and ∞. We distinguish two limiting cases: $(2k^2\psi)^{1/2} \to \infty$ and $(2k^2\psi)^{1/2} \to 0$; the autocorrelation function (2.10) is accordingly

$$\begin{aligned} (2k^2\psi \to \infty): R(t) \backsimeq \frac{A_0^2k^2\psi}{4\pi} \cos\omega_0 t\cdot\Big\{1 &+ \frac{\pi}{2} r_0(t) \\ &+ \sum_{n=0}^\infty \frac{r_0(t)^{2n+2}(2n)!}{2^{2n}n!^2(2n+2)(2n+1)}\Big\} + 0([2k^2\psi]^{1/2}), \end{aligned} \tag{2.15a}$$

$$\begin{aligned} = \frac{A_0^2k^2\psi}{4\pi} \cos\omega_0 t\cdot\Big\{&r_0(t)\Big[\frac{\pi}{2} + \sin^{-1} r_0\Big] \\ &+ (1 - r_0^2)^{1/2}\Big\} + 0([2k^2\psi]^{1/2}) \end{aligned} \tag{2.15b}$$

for the former, while in the latter instance we obtain as expected Eq. (2.5), with a correction term $o(e^{-\frac{1}{2}k^2\psi}nr_0(t)^3)$.

Equations (2.15) apply when the maximum amount of overmodulation, namely 50 per cent of the time, occurs, whereas (2.5) yields the correlation function when essentially no overmodulation takes place. Additional correction terms may be found in straightforward but tedious fashion. We note from Fig. (2.1) that the mean intensity spectrum is here more widely distributed about f_0 than for the case of ignorable over-modulation. The additional harmonics are (carrier $\times$ noise) noise products, stemming from the

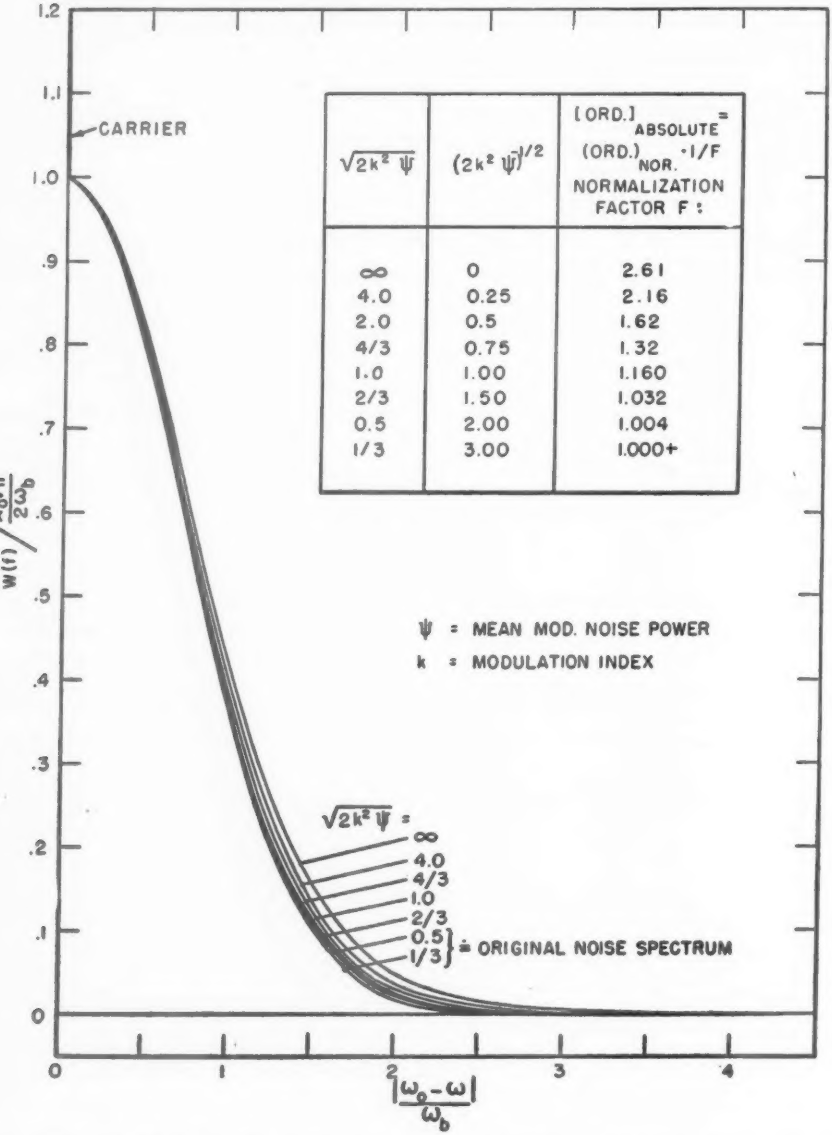

FIG. 2.1. Mean intensity spectrum of a carrier amplitude-modulated by noise.

clipping process inherent in overmodulation. However, [unlike the examples of frequency- or phase-modulation discussed earlier in ref. 1] there is a limit to the spread of the spectrum, determined by the fact that over-modulation can occur at the maximum but 50 per cent of the time. On the other hand, since the amount of clipping inherent in the weak modulation cases is ignorable, no significant spectral spread is obtained, (see Fig. 2.1). In any case the spectral spread is relatively small.

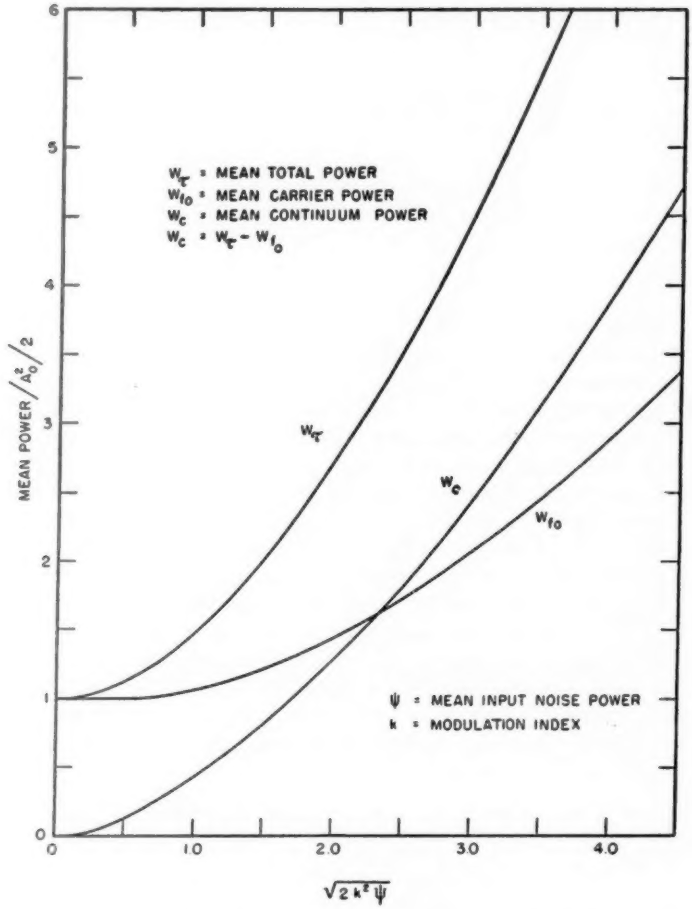

FIG. 2.2. Mean power in a carrier amplitude-modulated by noise.

Whereas the spectrum requires a series development, cf. (2.13), the total mean power W_τ and the total intensity W_c of the continuous part of the disturbance are easily obtained in precise, closed form from Eq. (2.2) if we remember that $kV_N(t) = y$ is normally distributed with the first-order probability density $W_1(y) = (2\pi k^2\psi)^{-1/2} \exp[-y^2/2k^2\psi]$.

The power in the carrier after modulation is (cf. (1.1) and (1.2))

$$
\begin{aligned}
W_{f_0} &= (A_0^2/2)\left|\int_{-1}^{\infty}(1+y)W_1(y)\,dy\right|^2 = (A_0^2/2)h_{0,0}^2 \\
&= (A_0^2/2)\{[1+\Theta([2k^2\psi]^{-1/2})]/2 - k^2\psi\phi^{(1)}([k^2\psi]^{-1/2})^2\}
\end{aligned} \tag{2.16}
$$

and

$$
\begin{aligned}
W_\tau &= R(0) = (A_0^2/2)\int_{-1}^{\infty}(1+y)^2W_1(y)\,dy \\
&= (A_0^2/2)\left[\left(\frac{1+k^2\psi}{2}\right)[\Theta([2k^2\psi]^{-1/2})+1]\right] - k^2\psi\phi^{(1)}([k^2\psi]^{-1/2})]
\end{aligned} \tag{2.17}
$$

and so

$$
\begin{aligned}
\therefore\ W_c = W_\tau - W_{f_0} &= (A_0^2/2)[1/4 + (k^2\psi/2)[\Theta\{(2k^2\psi)^{-1/2}\}+1] \\
&\qquad - \{\Theta[(2k^2\psi)^{-1/2}] - k^2\psi\phi([k^2\psi]^{-1/2})\}^2].
\end{aligned} \tag{2.18}
$$

Figure (2.2) illustrates the behavior of the mean intensities W_τ, W_{f_0}, W_c for different degrees of over-modulation. (See also Figs. 3.1 and 3.2 when $\mu = 0$.) Their limiting conditions are instructive: As the intensity of the modulating noise is increased, a correspondingly greater proportion of the modulated wave's power is distributed in the (*noise* $\times$ *noise*) noise sidebands, generated in the process of modulation and the clipping due to possible over-modulation, as shown in Fig. 2.2. We remark that the amount of power in the carrier component and in the continuum is quite independent of the particular spectral distribution of the original (normal random) noise, and depends only on the clipping level at which over-modulation occurs, since power in a given band of frequencies is proportional to the integrated intensity of the band. One can consider also more complicated modulations, such as square-law modulation, viz: $A_0(t) = [1 + kV_NI t)]^2$: the treatment is identical with that of the linear case examined here, following an appropriate modification of the transform relation (2.7).

3. Carrier amplitude-modulated by a signal and noise. The results of the previous section may be generalized to include the important case of modulation of a carrier, $A_0 \exp(i\omega_0 t)$, by a signal with an accompanying noise disturbance. The modulated carrier (2.1) may now be written

$$
\left.
\begin{aligned}
V(t) &= A_0(t)\exp(i\omega_0 t) \\
&= A_0[1 + kV_N(t) + \mu V_S(t)]\exp(i\omega_0 t), \qquad kV_N + \mu V_S \geq -1 \\
&= 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad kV_N + \mu V_S \leq -1
\end{aligned}
\right\} \tag{3.1}
$$

which includes possible over-modulation when the signal and noise are properly phased and sufficiently intense. As before, $A_0(t)$ is represented by a suitable contour integral [cf. (2.7)]. Remembering that $A_0(t)$ is a real quantity, we apply Eq. (3.1) and the results of section 2 to the general relation for the auto-correlation function of the modulated wave and obtain finally

$$R(t) = (1/2)\, Re\{V(t_0)V(t_0 + t)^*\}_{\text{stat.av.}}$$

$$= (A_0^2/2)\, Re\left\{\exp(-i\omega_0 t)(4\pi^2)^{-1} \int_C z^{-2} \exp(iz)\, dz \cdot \int_C \xi^{-2} \exp(i\xi) F_2(z, \xi; t)_N F_2(z, \xi; t)_S\, d\xi\right\}, \tag{3.2}$$

where $F_2(z, \xi; t)_N$ is the characteristic function for the accompanying noise, and

$$F_2(z, \xi; t)_S = [\exp(i\mu V_S(t_0)z + i\mu V_S(t_0 + t)\xi)]_{\text{stat.av.}} = T_0^{-1} \int_0^{T_0} \exp(i\mu V_S(t_0)z + i\mu V_S(t_0 + t)\xi)\, dt_0 \tag{3.3}$$

is the characteristic function for the signal, with T_0 the period of the modulation. Expanding the exponents in (3.3) in a double Fourier series and averaging over the period $T_0(=2\pi/\omega_a)$ we can write the signal's characteristic function in the general form

$$F_2(z, \xi; t)_S = \sum_{m=0}^{\infty} (-1)^m \epsilon_m B_m(z) B_m(\xi) \cos m\omega_a t. \tag{3.4}$$

For signals which are entirely stochastic, we may replace the time-average in (3.3) by its equivalent ensemble average, since we are assuming throughout stationary (ergodic) processes. An example of the latter type is provided when the modulating signal is a (normal) random noise, uncorrelated with the background interference, in which case

$$F_2(z, \xi; t)_S = \exp\{-\mu^2 \psi(0)_S (z^2 + \xi^2) - \mu^2 \psi(t)_S z\xi\},$$
$$\psi(t)_S = \psi(0)_S r_0(t)_S\,, \qquad (\psi(0)_S \equiv \psi_S). \tag{3.5}$$

Here $\psi(t)_S$ is the auto-correlation function of $V(t)_S$. The correlation function of the modulated wave is then given by (2.10), provided we replace $k^2\psi r_0(t)$ therein by $k^2\psi_N r_0(t)_N + \mu^2\psi_S r_0(t)_S$ and $k^2\psi$ in $h_{0,n}$, Eq. (2.11), by $k^2\psi_N + \mu^2\psi_S$. The resulting intensity spectrum is a superposition of $(n \times n)$ noise components, the carrier being the only periodic term.

Let us consider now the more complex situation involving a periodic signal. The auto-correlation function (3.2) becomes with the help of section 2 and (3.4)

$$R(t) = (A_0^2/2) \sum_{m=0}^{\infty} \epsilon_m (-1)^m [\cos(\omega_0 + m\omega_a)t + \cos(\omega_0 - m\omega_a)t] \cdot \frac{1}{2}\left\{h_{m,0}^2 + \sum_{n=2}^{\infty} \frac{(-1)^n (k^2\psi r_0(t))^n}{n!} h_{m,n}^2\right\} \tag{3.6}$$

where the amplitude functions $h_{m,n}$ are

$$h_{m,n} = (2\pi)^{-1} \int_C z^{n-2} B_m(z) \exp[iz - k^2\psi z^2/2]\, dz. \tag{3.7}$$

The mean intensity spectrum follows at once in the usual way with the aid of the theorem of Wiener and Khintchine, cf. (1.6). Because of the distortion inherent in over-modula-

tion we expect that the original signal will be deformed, and because of the accompanying background noise we may further predict that not only will there be cross-modulation between the components of the noise, but between them and the signal harmonics, modified in the process of over-modulation. Accordingly, we observe from (3.6) that the term in the correlation function for which $(m = 1, n = 0)$ corresponds to the carrier component, and the harmonics for which $(m \geq 1, n = 0)$ represent the $(s \times s)$ signal cross-terms generated in the course of over-modulation. On the other hand, the components for which $(m = 0, n \geq 1)$ are attributable to $(n \times n)$ noise harmonics, while for $(m \geq 1, n \geq 1)$ one has as the noise contribution $(s \times n)$ noise products. The mean power associated with the carrier, signal, and continuum are obtained from (3.6) on setting $t = \infty$ for the periodic components and $t = 0$ for the stochastic part of the modulated wave, according to section 1. Note that again, cf. section 2, the power content of the disturbance does not depend on the spectral distribution of the noise and signal modulations. (See also Appendix II of ref. [10].)

We can determine W_T by a more direct method than expansion in a double series, which involves one less infinite development, unlike (3.6). The procedure is based on the observation that $y = kV_N(t)$ and $z = \mu V_S(t)$ can be treated as independent random variables, whose first-order probability densities $W_1(y)$ and $w_1(z)$ are easily determined. Thus, for the background noise, $W_1(y)$ is a gauss distribution density for which $\langle y^2\rangle_{s.av.} = k^2\psi$, $\langle y\rangle_{s.av.} = 0$, while for a periodic signal

$$w_1(z) = \int_{-\infty}^{\infty} (2\pi)^{-1} \exp(iz\xi)\, d\xi \int_0^{2\pi} (2\pi)^{-1} \exp[i\mu\xi V_S(0, \phi)]\, d\phi. \tag{3.8}$$

The phase ϕ is a purely random quantity, distributed uniformly between 0 and 2π with a probability density $1/2\pi$. Therefore V_S is also a random variable, corresponding physically to the fact that we agree now to observe the periodic wave at (independent) random times. (In this fashion any periodic disturbance can be "randomized" with respect to the observer.) The mean-square value of the modulated carrier (3.1) becomes

$$\begin{aligned} W_T &= \langle | V(t_0) |^2\rangle_{s.av.}/2 = (A_0^2/2)\langle(1 + y + z)^2\rangle_{s.av.}\,, \text{ (for all } w = y + z \geq -1), \\ &= (A_0^2/2) \int_{-1}^{\infty} W_1^{(1)}(w)(1 + w)^2\, dw = R(0), \end{aligned} \tag{3.9}$$

where the frequency-function $W_1^{(1)}(w)$ for $w = y + z$ is given by

$$\begin{aligned} W_1^{(1)}(w) &= \int_{-\infty}^{\infty} W_1(y)w_1(w - y)\, dy \\ &= \int_{-\infty}^{\infty} (2\pi)^{-1} \exp(-iw\xi - k^2\psi\xi^2/2)\, d\xi \int_0^{2\pi} (2\pi)^{-1} \exp(i\mu\xi V_S(0, \phi))\, d\phi, \end{aligned} \tag{3.10}$$

since the Jacobian $| \partial(y, z)/\partial(y, w) |$ of the transformation is unity. In a similar way one obtains the mean power in the carrier, which is

$$\begin{aligned} W_{f_0} &= (A_0^2/2)h_{0,0}^2 = \langle | V(t) |\rangle_{s.av.}^2/2 = (A_0^2/2)\, | \langle 1 + y + z\rangle_{av.} |^2, \\ &\qquad\qquad \text{for all } w = y + z \geq -1 \\ &= (A_0^2/2)\left\{\int_{-1}^{\infty} W_1^{(1)}(w)(1 + w)\, dw\right\}^2, \end{aligned} \tag{3.11}$$

since w is real. Note that this procedure does *not* give the mean power $W_{\text{per.}}$ in the periodic components of the wave, but only the steady or average value of the envelope. To calculate $W_{\text{per.}}$ one must sum the series

$$\sum_{m=0}^{\infty} \epsilon_m (-1)^m h_{m,0}^2 \, .$$

In the specific case of a sinusoidal signal,

$$V_S = V_0 \cos(\omega_a t + \gamma),$$

we find from (3.3) and (3.4) that now $B_m(z) = J_m(\mu V_0 z)$, and so the amplitude functions (3.7) are explicitly*

$$h_{m,n} = (2\pi)^{-1} \int_C z^{n-2} J_m(\mu V_0 z) \exp(iz - k^2\psi z^2/2)\, dz$$

$$= \frac{-i^{m+n}}{2m!} \cdot \left(\frac{k^2\psi}{2}\right)^{(1-n)/2} \cdot [\mu V_0/(2k^2\psi)^{1/2}]^m \tag{3.12a}$$

$$\cdot \sum_{q=0}^{\infty} \frac{(2)^{q/2} {}_1F_1([q + m + n - 1]/2; m + 1; -\mu^2 V_0^2/2k^2\psi)}{q!(k^2\psi)^{q/2}\Gamma([3 - m - n - q]/2)},$$

or

$$= \frac{-i^{m+n}}{2} \cdot \left(\frac{k^2\psi}{2}\right)^{(1-n)/2} \cdot [\mu V_0/(2k^2\psi)^{1/2}]^m \sum_{q=0}^{\infty} \alpha_{mnq} (\mu^2 V_0^2/2k^2\psi)^q, \tag{3.12b}$$

where

$$\alpha_{mnq} = [1/q!(q + m)!] \left\{ \frac{{}_1F_1([2q + m + n - 1]/2; 1/2; -1/2k^2\psi)}{\Gamma([3 - 2q - m - n]/2)} \right.$$
$$\left. + (2/k^2\psi)^{1/2} \frac{{}_1F_1([2q + m + n]/2; 3/2; -1/2k^2\psi)}{\Gamma([2 - 2q - m - n]/2)} \right\}. \tag{3.12c}$$

The first result, (3.12a), is convenient when the signal is large relative to the noise $[\mu V_0^2 \gg 2k^2\psi]$, for then we may use the asymptotic form of the confluent hypergeometric function

$${}_1F_1(\alpha; \beta; -x) \backsimeq \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)} x^{-\alpha} \left\{ 1 + \frac{\alpha(\alpha - \beta + 1)}{x1!} \right.$$
$$\left. + \frac{\alpha(\alpha + 1)(\alpha - \beta + 1)(\alpha - \beta + 2)}{x^2 2!} + \cdots \right\}, \quad Re(x) > 0. \tag{3.13}$$

On the other hand, the series (3.12b) is particularly useful for weak signals $[\mu V_0^2 \ll 2k^2\psi]$. From either expression we can easily obtain the important limiting cases of overmodulation and weak noise $(2k^2\psi \to 0)$, the latter by (3.13).

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*The details of the integration are given in section 2 and in Appendix III of ref. [10]. See also Appendix IV of ref. [3].

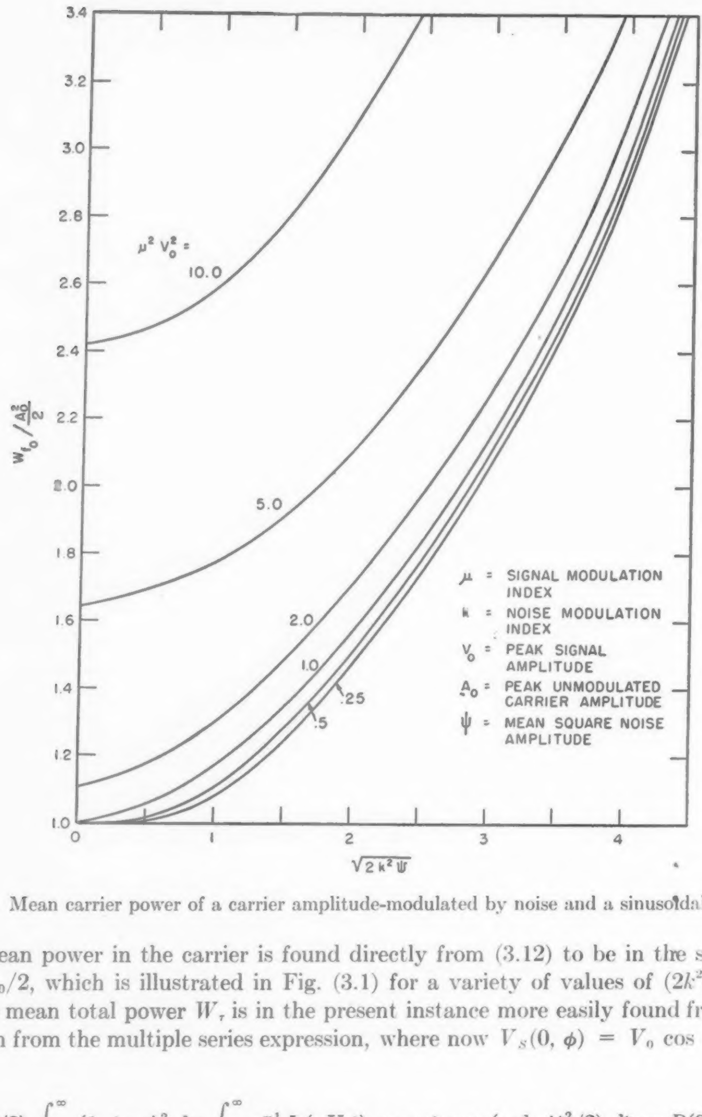

FIG. 3.1. Mean carrier power of a carrier amplitude-modulated by noise and a sinusoidal signal.

The mean power in the carrier is found directly from (3.12) to be in the sinusoidal case $A_0^2 h_{0,0}^2/2$, which is illustrated in Fig. (3.1) for a variety of values of $(2k^2\psi)^{1/2}$ and $\mu^2 V_0^2$. The mean total power W_τ is in the present instance more easily found from (3.8)-(3.10) than from the multiple series expression, where now $V_S(0, \phi) = V_0 \cos(\gamma + \phi)$. We have

$$W_\tau = (A_0^2/2) \int_{-1}^{\infty} (1 + w)^2 \, dw \int_0^{\infty} \pi^{-1} J_0(\mu V_0 \xi) \cos w\xi \exp(-k\ \psi \xi^2/2) \, d\xi = R(0) \tag{3.14a}$$

$$= (A_0^2/2)\left\{\left(\frac{1}{2} + \frac{k^2\psi}{2} + \frac{\mu^2 V_0^2}{4}\right)[1 + \Theta([2k^2\psi]^{-1/2})] - k^2\psi\phi^{(1)}[(k^2\psi)^{-1/2}] \right. \tag{3.14b}$$

$$\left. + k^2\psi \cdot \sum_{n=2}^{\infty} \left(\frac{\mu^2 V_0^2}{2k^2\psi}\right)^n \frac{1}{n!^2 2^n} \phi^{(2n-3)}[(k^2\psi)^{-1/2}]\right\}.$$

W_T is shown in Fig. (3.2) for representative values of $\mu^2 V_0^2$ and $(2k^2\psi)^{1/2}$. The power associated with the carrier part of the modulated wave is obtained alternatively from (3.11) in a similar manner; we have finally

$$W_{f_0} = (A_0^2/2)\Big\{[1 + \Theta\{(2k^2\psi)^{-1/2}\}]/2 - k^2\psi\phi^{(1)}[(k^2\psi)^{-1/2}] + (k^2\psi)^{1/2} \sum_{n=1}^{\infty} \left(\frac{\mu^2 V_0^2}{2k^2\psi}\right)^n \frac{1}{n!^2 2^n} \phi^{(2n-2)}[(k^2\psi)^{-1/2}]\Big\}^2 \tag{3.15}$$

which is equivalent to our result $A_0^2 h_{0,0}^2/2$. Typical curves for W_{f_0} as a function of $(2k^2\psi)^{1/2}$ are shown in Fig. (3.1). Since $W_{per} = W_{f_0} + W_{(s \times s)}$ one easily finds the mean power $W_{(s \times s)}$ associated with the signal components once W_{per} and W_{f_0} have been calculated. Furthermore, because $W_c = W_T - W_{per}$, we obtain also the mean power in the continuum, without having to sum the doubly-infinite series of the direct expansion (3.6).

For a given amount of signal (μV_0 fixed) the power in the periodic and in the carrier components of the modulated wave increases with the amount of modulating noise, as shown (for W_{f_0} only) in Fig. (3.1). The energy available in the signal becomes independent of the noise; however, the noise is then relatively so great that the signal is quite ignorable. This is easily seen from (3.12) in the case of the sinusoidal signal ($\backsim h_{1,0}^2$) when $2k^2\psi \to \infty$. Depending on the strength of the signal and noise relative to the amplitude A_0 of the unmodulated carrier, some of the remaining signal power is distributed in ($s \times s$) "discrete," periodic terms ($m \geq 2$, $n = 0$), which represent a distortion of the original sinusoid, attributable to the clipping inherent in over-modulation. Furthermore, as the noise becomes more intense, over-modulation occurs a significant fraction of the time, up to a maximum of 50 per cent. The amount of signal power in the modulated wave is then one-half that in the original modulating signal, since on the average the comparatively weak signal "rides" on the stronger noise half the time. Additional noise is also generated by the clipping of the wave, and these new noise components appear as ($s \times n$), ($m, n \geq 1$), and ($n \times n$), ($m = 0$, $n \geq 1$) terms, produced by the cross-modulation of the signal and noise harmonics. Only the ($n \times n$) terms are important when the noise is strong compared to the signal. In a similar way one finds that for weak noise and strong signals, the ($s \times n$) noise products are significant provided μV_0 is noticeably greater than unity (over-modulation of the carrier, due essentially to the signal). If μV_0 is less than unity and there is little noise, we expect a negligible amount of over-modulation, and the original signal consequently suffers no appreciable distortion. The same argument applies to describe the variation of the mean-total power W_T and the mean continuum power $W_c = W_T - W_{per}$ with different amounts of signal and noise modulation. Again, when the noise is the dominant factor, both W_T and W_c become indefinitely large as $2k^2\psi \to \infty$, cf. Fig. (3.2). We note also that W_{per} approaches a fixed limit, independent of the amount of modulating noise, which is one-half the original power in the carrier and modulating signal. Most of the wave's energy goes now into the noise continuum ($s \times n$, $n \times n$), as a result of the very heavy ($\backsim$50 per cent) over-modulation. When the noise becomes weaker, correspondingly less of the modulated carrier's power is distributed in the continuum. In general, different degrees of modulation (i.e., different values of μ and k) will, as expected, critically affect the magnitudes of total, periodic, and continuum powers.

The explicit calculation of spectra, based as it must be on Eq. (3.6), is far more

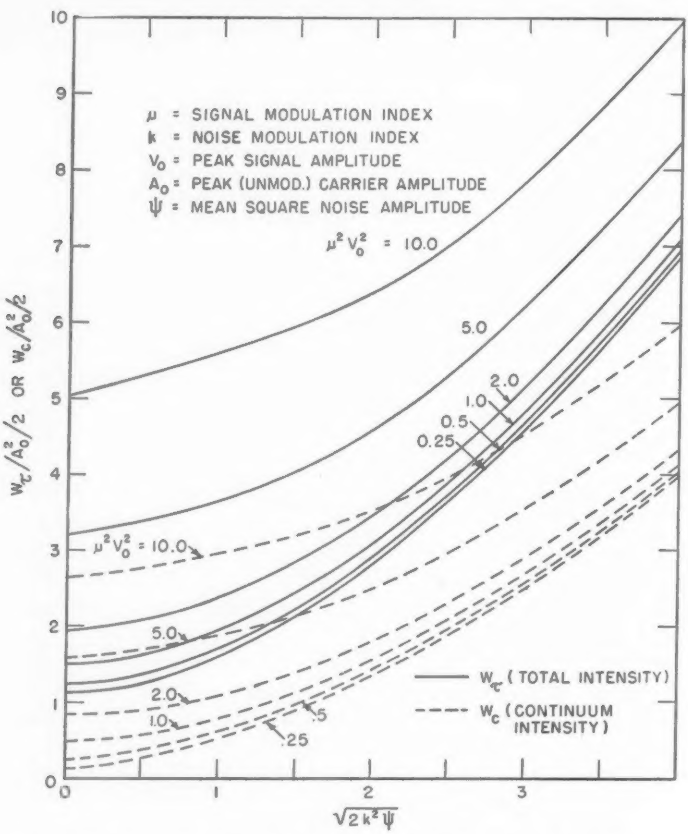

FIG. 3.2. Mean total and continuum powers in a carrier amplitude-modulated by noise and a sinusoidal signal

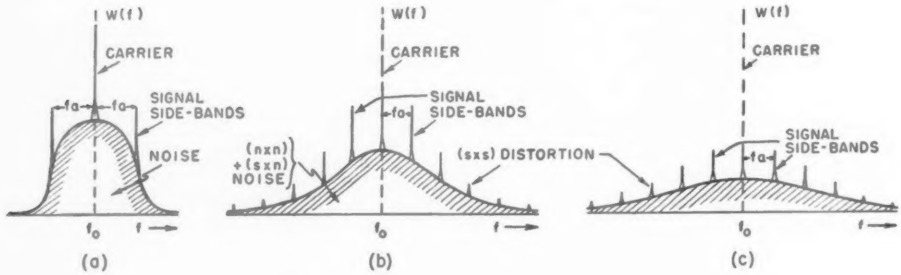

FIG. 3.3. Spectra:
(a) No over-modulation,
(b) Moderate over-modulation,
(c) 50% over-modulation.

tedious than the determination of power. A sketch illustrating typical spectra is given in Fig. (3.3); the precise calculations are reserved for a later paper.

With the help of the properties of the hypergeometric function one can obtain other interesting limiting expressions for the power and the spectrum.* For example, when there is ignorable over-modulation, $(\mu^2\langle V_S^2\rangle_{s.av.} + k^2\langle V_N^2\rangle_{s.av.} \ll 1)$, $h_{m,n}$, $(m \geq 2, n \geq 1)$, approaches zero, and one has as expected

$$R(t) \doteq (A_0^2/2)\{1 + k^2 r_0(t)\psi + \mu^2\langle V_S(t_0)V_S(t_0 + t)\rangle_{av.}\} \cos \omega_0 t, \qquad (\mu^2\langle V_S^2\rangle_{s.av.} + k^2\langle V_N^2\rangle_{s.av.} \ll 1). \tag{3.16}$$

For no signal at all the results of the preceding section can be applied, while at the other extreme of no modulating noise ($\psi = 0$) one finds $R(t)$ from (3.6) and (3.7) on setting $n = 0$' In the specific case of sinusoidal modulation the amplitude functions are** now

$$h_{m,0} = \int_C J_m(\mu V_0 z) \exp(iz)\, dz/2\pi z^2$$

$$= \frac{i^m \mu V_0}{4} \cdot \left\{ \frac{{}_2F_1([m-1]/2, [-m-1]/2; 1/2; 1/\mu^2 V_0^2)}{\Gamma([3+m]/2)\Gamma([3-m]/2)} + \frac{2}{\mu V_0} \frac{{}_2F_1(m/2, -m/2; 3/2; 1/\mu^2 V_0^2)}{\Gamma([2+m]/2)\Gamma([2-m]/2)} \right\}, \qquad (1 \leq \mu V_0) \tag{3.17a}$$

$$= \left.\begin{array}{ll} -1\;, & m = 0, \\ i\mu V_0/2, & m = 1, \\ 0, & m \geq 2, \end{array}\right\}, \qquad (0 \leq \mu V_0 \leq 1). \tag{3.17b}$$

The mean total power is found from (3.14a) to be, $(k^2\psi = 0)$,

$$W_r = R(0) = (A_0^2/2\pi) \int_{-z_0}^{1} \frac{(\mu V_0 z + 1)^2}{(1 - z^2)^{1/2}}\, dz, \qquad z_0 = \begin{cases} 1/\mu V_0\,, & \mu V_0 \geq 1 \\ 1, & 0 \leq \mu V_0 \leq 1 \end{cases}$$

$$= (A_0^2/2)\left\{\left(\frac{1}{2} + \frac{1}{\pi}\sin^{-1} z_0\right)\left(1 + \frac{\mu^2 V_0^2}{2}\right) + \frac{2\mu V_0}{2}\left(1 - \frac{\mu V_0 z_0}{4}\right)(1 - z_0^2)^{1/2}\right\} \tag{3.18}$$

while the mean power in the carrier is

$$W_{f_0} = (A_0^2/2) \left| \int_{-z_0}^{1} \frac{(\mu V_0 z + 1)}{\pi(1 - z^2)^{1/2}}\, dz \right|^2$$

$$= (A_0^2/2)\left\{\frac{1}{2} + \frac{1}{\pi}\sin^{-1} z_0 + \frac{\mu V_0}{\pi}(1 - z_0^2)^{1/2}\right\}^2, \qquad (2k^2\psi \to 0). \tag{3.19}$$

The difference $W_r - W_{f_0}$ now represents the mean power in the continuum, in this case

*See section 4 of ref. [10]; in particular the case of the biased, ν th—law rectifier.

**See Eqs. (4.19), (4.20), and (4.22) of ref. 10.

the discrete spectra consisting of the harmonics of the modulation and its distortion (if any) due to over-modulation. When there is no signal, only terms in (3.6) and (3.7) for which $m = 0$ remain, and our general expressions of the present section reduce to the results of the preceding section, cf. Eq. (3.16) et. seq. Other limiting cases may be treated in the manner of section 4, reference [10].

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ON THE NUMERICAL SOLUTION OF THE DIRICHLET PROBLEM FOR LAPLACE'S DIFFERENCE EQUATION*

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1. Introduction. The present note is concerned with the proof of the convergence of three iterative methods for the numerical solution of the difference equation boundary value problem which is analogous to the classical Dirichlet problem for Laplace's differential equation. These three iterative methods for the Dirichlet difference equation problem bear a marked resemblance to, and are patterned after, methods of solution for the Dirichlet differential equation problem, and may be briefly characterized as follows: (a) method I is the precise analogue of Poincaré's [1][1] "methode de balayage", (b) method II is the precise analogue of the extension of Poincaré' method due to Kellogg [2], (c) method III is based on a certain connection between Laplace's differential equation and the heat equation (intuitively put, "the steady state temperature is a harmonic function")'

Several convergence proofs for the Dirichlet difference boundary value problem have been based on Liebmann's [3] original proof (see P. Frank and R. von Mises [4], and I. G. Petrowsky [5]) which involves a definite ordering of the points of the domain. It follows from the convergence proof given below for method III that this preliminary ordering of the points of the domain is unnecessary. R. Courant [6], has given a convergence proof for method I for the Dirichlet difference boundary value problem employing a variational method which is the direct analogue of the classical Dirichlet's principle for Dirichlet's differential boundary value problem. In spite of the inherent interest and symmetry of the proofs in methods I and II described below, it should be remarked that the variational method of proof (minimization of a quadratic sum) proposed by Courant possesses a certain advantage in that it may be applied to other "elliptic" difference equation boundary value problems, since it does not require for its application the maximum-minimum property of the solutions of the difference equation, i.e. that "the maximum and the minimum of a solution of the difference equation occurs on the boundary". In the Laplace case, both the differential and the difference equations possess the maximum-minimum property. However, it is easy to give examples of elliptic differential equations (for which the maximum-minimum principle holds) e. g. $U_{xx} + U_{xy} + U_{yy} = 0$, which have the property that the seemingly most naturally associated difference equation does not possess the maximum-minimum property. A convergence proof for method III, different from the one given here, has been given by Feller [7].

Although only the two dimensional case will be considered here explicitly, it is clear that the same arguments are valid in a finite number of dimensions.

2. Convergence proofs. The boundary value problem will be formulated first. Consider a finite set S of points (m, n) in the xy-plane, where m and n are integers (i.e., S is "a net of equally space points in the plane, where for convenience the spacing has

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[1]Numbers in brackets refer to the bibliography at the end of the paper.

been taken as unity"). The points of the set S are divided into two mutually exclusive subsets of points, the subset D consisting of interior points of S and the subset C of boundary points of S. A point (m, n) of S is said to be an interior point of S provided that its "four neighbors" $(m \pm 1, n)$, $(m, n \pm 1)$ also belong to S. A point (m, n) of S is said to be a boundary point of S provided that at least one of its four neighbors $(m \pm 1, n)$, $(m, n \pm 1)$ does not belong to S. It will be supposed in what follows that every point of the boundary C has at least one neighbor in D (this is clearly no restriction, as far as the difference equation problem is concerned, since only "isolated" points of the set S and "unessential" boundary points of S are excluded).

The Dirichlet difference boundary value problem for the finite set $S = D + C$ consists in finding a real valued function u, defined on $D + C$, assuming prescribed values on the boundary C and which satisfies Laplace's difference equation

$$4u(m, n) = u(m + 1, n) + u(m - 1, n) + u(m, n + 1) + u(m, n - 1), \tag{1}$$

for each point (m, n) in the interior D.

This boundary value problem, which is analogous to the classical Dirichlet problem, is well known to possess one and only one solution. This is readily seen as follows. The difference equation (1) amounts to a system of d linear equations in the d unknown values of u at the points of D, supposed to be d in number. This system of linear equations is homogeneous if and only if the prescribed values of u on C are all zero (recall that it was assumed that every point of C has at least one neighbor in D). But it is known from algebra that a system of d linear equations in d unknowns will have a unique solution if and only if the corresponding homogeneous system has only the trivial, identically zero solution. That the homogeneous system has only the zero solution follows immediately once it is shown that the maximum and minimum values of a function u, defined on $D + C$, and satisfying (1) in D, must occur on the boundary C. Suppose, on the contrary, that the maximum value M of u occurs at a point (m, n) of D, but not on the boundary C. Then from (1) it follows that the value of u at the four points $(m \pm 1, n)$, $(m, n \pm 1)$ must also be M and hence these four points must all be in the interior, D. By induction, it follows that the infinite set of points $(m + j, n)$, $j = 0, 1, 2, \cdots$ all belong to D, which contradicts the initial assumption that D is a finite set of points. Thus the maximum value M must occur at a point of C. Since a minimum of the function u is a maximum of the function $-u$, and $-u$ also satisfies (1), it follows that the minimum of u on $D + C$ also occurs on C.

Method I.

Consider the convergence of method I mentioned in the introduction. Denote by f the given real valued function, defined on C, which is the prescribed boundary value of u on C, and let the "initial function" G be a superharmonic function defined on $D + C$ and which coincides with the given function f on the boundary C. That is

$$4G(m, n) \geqq G(m + 1, n) + G(m - 1, n) + G(m, n + 1) + G(m, n - 1), \tag{2}$$

for each (m, n) in D, and

$$G(m, n) = f(m, n),$$

for each (m, n) in C. The restriction that the initial function G be superharmonic is not essential, and will be removed later.

Let the points of D be arranged in a sequence $P_1, P_2, P_3, \cdots$ in such a way that each point of D occurs infinitely many times in the sequence. For convenience, the sequence may be chosen by moving from point to point of D in some definite geometrical pattern, but this is not necessary.

A sequence of functions is now defined on $D + C$ by the following procedure. First, let $w_0 \equiv G$, and define the sequence of functions $w_0, w_1, w_2, \cdots$ successively as follows:

First step: $w_0 \equiv G$, on $D + C$;

Second step: w_1 is harmonic at P_1,
$w_1 = w_0$ on $D + C - P_1$;

Third step: w_2 is harmonic at P_2,
$w_2 = w_1$ on $D + C - P_2$;

.

.

.

$(p + 1)$ st. step: w_p is harmonic at P_p,
$w_p = w_{p-1}$ on $D + C - P_p$;

etc. In other words, one moves from point to point in D changing the value of the functions at each point so that equation (1) holds.

It is easily seen that each function w_p is superharmonic in $D + C$. Consider w_1, and let $P_1 = (m, n)$. w_1 is certainly superharmonic at all points of $D + C$, save perhaps at the five points (m, n), $(m \pm 1, n)$, $(m, n \pm 1)$. But at (m, n) the equality sign holds in equation (2), with G replaced by w_1, by the definition of $w_1(m, n)$, while *at* the four points $(m \pm 1, n)$, $(m, n \pm 1)$, equation (2), with G replaced by w_1, holds a fortiori, since the right hand sum does not increase if G is replaced by w_1. By induction, w_p is superharmonic for each p. Further, the sequence of functions $w_0, w_1, w_2, \cdots$ is monotonically non-increasing on $D + C$, by the way in which the sequence was constructed. Finally, each function w_p is bounded below by the minimum value of G on $D + C$.

This last statement follows immediately from the fact that the minimum value of a function superharmonic in $D + C$ must occur at a point of C (this may be seen by an argument analogous to that used above in proving the maximum-minimum property of solutions of (1)). Since

$$w_p(m, n) = G(m, n) = f(m, n),$$

for (m, n) in C and any p, it follows that for any p

$$w_p(m, n) \geqq \min_C G = \min_C f,$$

for any (m, n) in D; and this, together with

$$w_1(m, n) \geqq w_2(m, n) \geqq \cdots \geqq w_p(m, n) \geqq \cdots$$

assures the convergence of the sequence w_p at each point of D. It only remains to show that the limit function

$$\lim_{p \to \infty} w_p(m, n),$$

(m, n) in $D + C$, is the solution of the boundary value problem.

There is no question about the fact that the limit function coincides with f on C, since each w_p does, it is only necessary to show that the limit function satisfies the partial difference equation (1) in D. Consider a point (m, n) of D. Since (m, n) occurs infinitely many times in the sequence $P_1, P_2, P_3, \cdots$ it follows that there is an infinite sequence of integers, k_p, such that

$$(m, n) = P_{k_p},$$

for $p = 1, 2, 3, \cdots$. Consequently, there is a subsequence of functions $w_{k_1}, w_{k_2}, \cdots, w_{k_p}, \cdots$ of the sequence $w_1, w_2, \cdots, w_p, \cdots$ such that each function w_{k_p} of the subsequence is harmonic at (m, n), that is

$$4w_{k_p}(m, n) = w_{k_p}(m + 1, n) + w_{k_p}(m - 1, n) + w_{k_p}(m, n + 1) + w_{k_p}(m, n - 1).$$

Since

$$\lim_{p\to\infty} w_p(x, y) = \lim_{p\to\infty} w_{k_p}(x, y),$$

for *any* (x, y) in D, it follows that

$$4 \lim_{p\to\infty} w_p(m, n) = \lim_{p\to\infty} w_p(m + 1, n) + \lim_{p\to\infty} w_p(m - 1, n) + \lim_{p\to\infty} w_p(m, n + 1) + \lim_{p\to\infty} w_p(m, n - 1),$$

i.e., the limit function is a solution of the boundary value problem (notice that the existence of a solution has been proved independently of the earlier considerations involving Cramer's rule). The solution has already been shown to be unique by the maximum-minimum principle.

The restriction that the initial function G be superharmonic will now be removed by showing that any function defined on $D + C$ may always be represented as the difference of two superharmonic functions on $D + C$. Let g be any function defined on $D + C$, and let M be one fourth of the maximum of the absolute value of the Laplacian of g on D, i.e. $M = \frac{1}{4} \max_D | g(m + 1, n) + g(m - 1, n) + g(m, n + 1) + g(m, n - 1) - 4g(m, n) |$.

Then

$$g(m, n) = g_1(m, n) + g_2(m, n),$$

where

$$g_1(m, n) = g(m, n) - M(m^2 + n^2),$$

$$g_2(m, n) = -M(m^2 + n^2).$$

Clearly, g_1 and g_2 are both superharmonic on $D + C$. Thus, if g is any function taking the prescribed values on C, the sequences of functions, constructed as explained above, corresponding to the initial superharmonic functions g_1 and g_2 converge to harmonic functions, and the difference of the two limit functions is the desired solution.

The method just described is the precise analogue of Poincaré's [1] "method of sweeping out" for the classical Dirichlet problem. It also coincides with the relaxation method (Southwell [8]) provided that the relaxation is done point by point and that at each point the residual is actually reduced to zero.

Method II.

This method differs from the preceding one only in the removal of the restriction that the process of method I be carried out pointwise at each step. Consider a sequence $B_1, B_2, B_3, \cdots$ of subregions ("blocks") of D, subject to the following conditions: (a) each point of D is an interior point of an infinite number of the blocks of the sequence, (b) the Dirichlet problem is explicitly solvable for each block B_p in terms of arbitrary boundary values on the boundary of B_p.

The convergence proof is the same as in method I. One need only substitute the sequence $B_1, B_2, B_3, \cdots$ of blocks for the sequence $P_1, P_2, P_3, \cdots$ of points.

This method is the exact analogue of the procedure suggested for the classical Dirichlet problem by Kellogg [2]. It is not to be confused with "block relaxation", with which it coincides only in the case where the residuals are actually made zero at each interior point of each block.

Method III.

Method III is essentially a method of successive approximations. Starting with an initial function $w_0 = g$ which satisfies the prescribed boundary condition, one defines the following sequence of functions w_p:

$$\begin{aligned}
w_0 &= g, \text{ on } D + C;\\
4w_1(m, n) &= w_0(m+1, n) + w_0(m-1, n) + w_0(m, n+1)\\
&\qquad + w_0(m, n-1), \text{ for } (m, n) \text{ in } D,\\
w_1(m, n) &= g(m, n), \text{ for } (m, n) \text{ in } C;\\
&\ \ \vdots\\
4w_p(m, n) &= w_{p-1}(m+1, n) + w_{p-1}(m-1, n) + w_{p-1}(m, n+1)\\
&\qquad + w_{p-1}(m, n-1), \text{ for } (m, n) \text{ in } D,\\
w_p(m, n) &= g(m, n), \text{ for } (m, n) \text{ in } C;\\
&\ \ \vdots
\end{aligned}$$

Again one may suppose, without loss of generality, that the initial function g is superharmonic to start out with. Hence all the functions w_p are superharmonic and from the minimum principle for superharmonic functions it follows that

$$w_0 \geqq w_1 \geqq w_2 \geqq \cdots \geqq w_p \geqq \cdots \geqq \min_C g = \min_C f.$$

Thus the sequence converges, and since

$$\lim_{p\to\infty} w_p = \lim_{p\to\infty} w_{p-1},$$

it follows that the limit function of the sequence is the desired solution of the Dirichlet problem.

The last method is related to the difference equation formulation of the heat flow problem, Emmons [9], in which the boundary C is held at all times at a fixed temperature, g represents the initial temperature, and the functions w_p represent the temperature of $D + C$ at regular intervals of time after $t = 0$. The limit function represents the steady state temperature. A proof of the convergence of method III, based upon random walk considerations, has been given by Feller [7].

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ON THE NON-LINEAR VIBRATION OF ELASTIC BARS*

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1. Introduction. The classical theory of vibration of bars is based on certain restrictive assumptions, namely: (a) the deflection is small; (b) supports are free to move in the axial direction; (c) deflection is inextensional. In practice, however, very often some or all of these assumptions are violated. Therefore, it is necessary to reformulate the problem of vibration of bars in its general form without these assumptions so that the domain of applicability of the classical theory can be well defined and problems to which the classical theory is not applicable can be attacked.

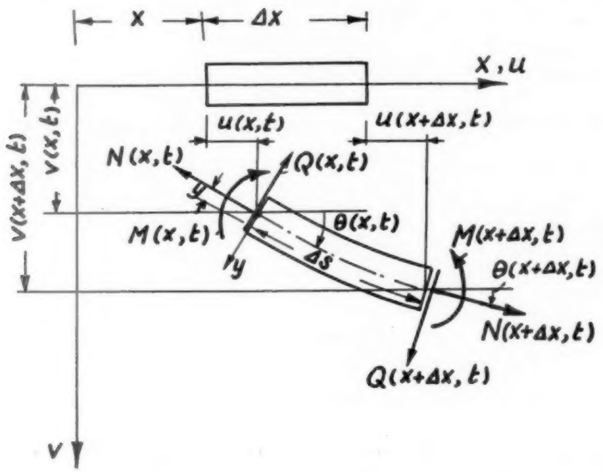

FIG. 1. Displaced element of beam.

Recently Woinowsky-Krieger [1] studied the effect of axial force on the vibration of hinged bars. N. J. Hoff [2] gave an analysis for the effect of inertia forces on the buckling of columns. In both of these analyses the classical treatment was improved by consideration of the axial stress due to bending.

In the following analysis, first the problem of vibration of bars is reformulated without any of the above-mentioned assumptions. Following that, a basic problem, free vibration of elastic bars having immovable hinged ends, is solved with the use of the perturbation method. The solution of this problem adequately describes those motions in which the changes in axial tension, as well as in deflection, are not small.

Solutions of vibration problems concerning bars with other types of end conditions and forced vibrations are left to a forthcoming paper.

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2. Equations of motion. The equations of dynamic equilibrium of an element of the beam deformed to a plane figure are (Fig. 1)

$$\begin{gathered} -\rho A u_{,tt} + (N \cos \theta)_{,x} - (Q \sin \theta)_{,x} = 0, \\ -\rho A v_{,tt} + (N \sin \theta)_{,x} + (Q \cos \theta)_{,x} = 0, \\ (1 + u_{,x}) J\theta_{,tt} + M_{,x} - Q[(1 + u_{,x}) \cos \theta + v_{,x} \sin \theta] \\ + N[v_{,x} \cos \theta - (1 + u_{,x}) \sin \theta] = 0. \end{gathered} \tag{1}$$

where ρ is the mass per unit volume, A the cross-sectional area of the beam, J the mass moment inertia per unit length, N, Q, M are the axial force, the shear force, and the bending moment at a point x, u and v are the deflections of any point x in the axial and transverse directions, θ is the angle between the tangent to the median line and the x axis, and t is the time. In calculating the inertia forces, the effect of changes of mass during motion is neglected.

The length element, ds, after deformation is given by

$$ds^2 = \{[v_{,x} + (y \cos \theta)_{,x}]^2 + [1 + u_{,x} - (y \sin \theta)_{,x}]^2\}\, dx^2, \quad \tan \theta = v_{,x}/(1 + u_{,x}). \tag{2}$$

Indices after a comma represent differentiation. In deriving Eq. (2) the effect of shear deformation is neglected.

Normal strain is defined as

$$e = \frac{ds}{dx} - 1 \tag{3}$$

Equations (2) and (3) are combined to yield

$$e = \epsilon - y\theta_{,x}\,, \qquad \epsilon = \frac{1 + u_{,x}}{\cos \theta} - 1 = \frac{v_{,x}}{\sin \theta} - 1. \tag{4}$$

Here, ϵ is the strain referred to the median line of the undeformed beam.

Hooke's Law states that

$$\sigma = Ee. \tag{5}$$

Thus,

$$N = \int \sigma\, dA = EA\epsilon, \qquad M = \int \sigma y\, dA = -EI\theta_{,x} \tag{6}$$

are calculated with the use of the first Eq. (4). Here I is the moment inertia of any section about its neutral axis. The variation of I with time is neglected.

Equations (6) and the third Eq. (1) are combined to give

$$Q = -\frac{(EI\theta_{,x})_{,x}}{1 + \epsilon} + J\theta_{,tt} \cos \theta. \tag{7}$$

The first two Eqs. (1) can be transformed into more suitable form by introducing non-dimensional quantities and multiplying the second by $i = \sqrt{-1}$ and adding the result to the first. The resulting equation is differentiated with respect to x. Thus,

$$
\begin{gathered}
-[(1+\epsilon)e^{i\theta}]_{,\tau\tau} + \left\{\left[\epsilon + i\lambda^2\left(-\frac{\theta_{,yy}}{1+\epsilon} + \theta_{,\tau\tau}\cos\theta\right)\right]e^{i\theta}\right\}_{,yy} = 0, \\
\tau = (E/\rho L^2)^{1/2}t, \qquad y = x/L, \qquad \lambda^2 = J/\rho A L^2.
\end{gathered} \tag{8}
$$

The second Eq. (4) may be transformed to

$$
y + \frac{1}{L}\,[u(x, t) + iv(x, t) - u(0, t) - iv(0, t)] = \int_0^y (1+\epsilon)e^{i\theta}\,dy. \tag{9}
$$

The first bracket in the first Eq. (8) represents the effect of translational inertia; the first, second, and third terms in the brace are respectively the contributions of the extension of the median line, the bending, and the rotatory inertia.

For $\lambda = 0$ the first Eq. (8) reduces to the one obtained by Carrier [4] for the problem of non-linear vibration of the elastic string, when ϵ is replaced by $(\epsilon - \epsilon_0)$ in order to include the initial tension.

Further, let

$$
s = \lambda\tau = t(EIg/WL^3)^{1/2}, \tag{10}
$$

where W is the total weight of the beam. Equations (8), when separated into real and imaginary parts, become

$$
\begin{gathered}
-\lambda^2\epsilon_{,ss} + (1+\epsilon)\lambda^2\theta_{,s}^2 + \epsilon_{,yy} - \epsilon\theta_{,y}^2 + \frac{2\lambda^2\theta_{,y}\theta_{,yyy}}{1+\epsilon} - 2\lambda^2\frac{\epsilon_{,y}\theta_{,y}\theta_{,yy}}{(1+\epsilon)^2} \\
+ \lambda^2\frac{\theta_{,yy}^2}{1+\epsilon} - 2\lambda^4\theta_{,ssy}\theta_{,y}\cos\theta + 2\lambda^4\theta_{,ss}\theta_{,y}^2\sin\theta - \lambda^4\theta_{,ss}\theta_{,yy}\cos\theta = 0,
\end{gathered} \tag{11}
$$

$$
\begin{gathered}
-2\lambda^2\epsilon_{,s}\theta_{,s} - (1+\epsilon)\lambda^2\theta_{,ss} + 2\epsilon_{,y}\theta_{,y} + \epsilon\theta_{,yy} - \lambda^2\frac{\theta_{,yyyy}}{1+\epsilon} + 2\lambda^2\frac{\epsilon_{,y}\theta_{,yyy}}{(1+\epsilon)^2} \\
+ \lambda^2\frac{\epsilon_{,yy}\theta_{,yy}}{(1+\epsilon)^2} - 2\lambda^2\frac{\epsilon_{,y}^2\theta_{,yy}}{(1+\epsilon)^3} + \lambda^2\frac{\theta_{,y}^2\theta_{,yy}}{1+\epsilon} + \lambda^4\theta_{,ssyy}\cos\theta - 2\lambda^4\theta_{,y}\theta_{,ssy}\sin\theta \\
- 2\lambda^4\theta_{,y}^2\theta_{,ss}\cos\theta - \lambda^4\theta_{,ss}\theta_{,yy}\sin\theta = 0.
\end{gathered} \tag{12}
$$

Contrary to expectation, it can be seen that $\epsilon = 0$ is not a possible solution since Eqs. (11) and (12) reduce to two independent equations for θ. Consequently, assumption (c) of the classical theory is not valid. It can be seen, however, that this assumption will be valid when ϵ is a higher-order quantity than θ. Therefore, the classical theory is obtained as a limiting case of the present theory for $\theta \to 0$.

3. Beams with immovable hinged ends. The boundary conditions for simply-supported beams with immovable hinged ends are

$$
\theta_{,y} = 0, \qquad u = v = 0 \qquad \text{for } y = 0, 1. \tag{13}
$$

The perturbation parameter is chosen as λ which represents the ratio of rotational inertia to translational inertia. Reversal of the sign of λ should reverse the sign of θ but not ϵ; θ and ϵ are therefore expanded into power series of λ as follows

$$
\theta = \lambda\theta_1 + \lambda^3\theta_3 + \lambda^5\theta_5 + \cdots, \qquad \epsilon = \lambda^2\epsilon_2 + \lambda^4\epsilon_4 + \cdots. \tag{14}
$$

The expressions for θ and ϵ are substituted into Eqs. (11) and (12) to obtain differential equations and into (13) and (9) to obtain boundary conditions.

Differential equations

$$(\lambda^2): \quad \epsilon_{2,yy} = 0,$$

$$(\lambda^4): \quad \epsilon_{4,yy} = \epsilon_{2,ss} - \theta_{1,s}^2 - 2\theta_{1,y}\theta_{1,yyy} - \theta_{1,yy}^2 - 2\epsilon_2\epsilon_{2,yy} + \epsilon_2\theta_{1,y}^2 ,$$

$$(\lambda^6): \quad \cdots , \tag{15}$$

$$(\lambda^3): \quad \theta_{1,ss} + \theta_{1,yyyy} - \epsilon_2\theta_{1,yy} - 2\epsilon_{2,y}\theta_{1,y} = 0,$$

$$(\lambda^5): \quad \theta_{3,ss} + \theta_{3,yyyy} - \epsilon_2\theta_{3,yy} = \theta_{1,ssyy} - 2\epsilon_{2,s}\theta_{1,s} - 4\epsilon_2\theta_{1,ss}$$

$$- 2\epsilon_2\theta_{1,yyyy} + 2\epsilon_{2,y}\theta_{1,yyy} + \epsilon_{2,yy}\theta_{1,yy} + \theta_{1,y}^2\theta_{1,yy} + 2\epsilon_{2,y}\theta_{3,y}$$

$$+ 2\epsilon_{4,y}\theta_{1,y} + 6\epsilon_2\epsilon_{2,y}\theta_{1,y} + \epsilon_4\theta_{1,yy} + 3\epsilon_2^2\theta_{1,yy} = 0,$$

$$(\lambda^7): \quad \cdots ; \tag{16}$$

Boundary conditions

$$\text{at} \quad y = 0, 1: \quad \theta_{i,y} = 0, \quad (i = 1, 2, \cdots) \tag{17}$$

$$(\lambda): \int_0^1 \theta_1 \, dy = 0, \qquad (\lambda^2): \int_0^1 \left(\epsilon_2 - \frac{\theta_1^2}{2}\right) dy = 0,$$

$$(\lambda^3): \int_0^1 \left(\epsilon_2\theta_1 + \theta_3 - \frac{1}{6}\,\theta_1^3\right) dy = 0, \quad (\lambda^4): \int_0^1 \left(\epsilon_4 - \epsilon_2\frac{\theta_1^2}{2} - \theta_1\theta_3 + \frac{\theta_1^4}{24}\right) dy = 0,$$

$$(\lambda^5): \cdots , \qquad (18), \quad (\lambda^6): \cdots ; \tag{19}$$

Initial conditions

$$\text{at} \quad s = 0: \quad \theta_1 = m\pi\theta_0 \cos m\pi y, \quad \theta_j = 0, \quad \theta_{i,s} = 0, \quad \epsilon_{i,s} = 0,$$

$$(i = 1, 2, \cdots ; j \neq 1) \tag{20}$$

The solution of the first Eq. (15) under the condition that $\epsilon_2(0, s) = \epsilon_2(1, s)$ and the first Eq. (19) is

$$\epsilon_2 = \frac{1}{2}\int_0^1 \theta_1^2 \, dy. \tag{21}$$

Substitution of (21) into the first Eq. (16) leads to

$$\theta_{1,ss} + \theta_{1,yyyy} - \frac{1}{2}\,\theta_{1,yy}\int_0^1 \theta_1^2 \, dy = 0. \tag{22}$$

Equation (22) is solved under the boundary conditions (17) and the first Eq. (18) for an initial sinusoidal deflection given by the first Eq. (20).

Let

$$\theta_1 = m\pi\theta_0 \cdot S(s) \cos m\pi y, \qquad S(0) = 1. \tag{23}$$

Substitution of Eq. (23) into Eq. (22) gives the following differential equation:

$$S_{,ss} + (m\pi)^4 S + \tfrac{1}{4}(m\pi)^4\theta_0^2 S^3 = 0. \tag{24}$$

Multiplying (24) by $S_{,s}$, a first integral can be obtained immediately. The resulting equation is of first order and separable. Integration can be effected in terms of elliptic functions. The result is

$$\begin{aligned} \theta_1 &= m\pi\theta_0 \cos m\pi y \operatorname{cn}(\omega_1 s, k), & \omega_1 &= m^2\pi^2\left(1 + \frac{1}{4}\theta_0^2\right)^{1/2}, \\ k^2 &= \left(2 + \frac{8}{\theta_0^2}\right)^{-1}, & \frac{T_1}{T_0} &= \frac{2}{\pi}\left(1 + \frac{\theta_0^2}{4}\right)^{-1/2} K, \end{aligned} \tag{25}$$

where cn and K denote the elliptic cosine and the complete elliptic integral of the first kind; T_1/T_0 is the ratio of non-linear period to linear period. The use of Eq. (21) gives

$$\epsilon_2 = \tfrac{1}{4}(m\pi\theta_0)^2 \operatorname{cn}^2(\omega_1 s, k). \tag{26}$$

Substitution of (25) and (26) into the second Eq. (15), integration, and the condition $\epsilon_4(0, s) = \epsilon_4(1, s)$, yield

$$\begin{aligned} \epsilon_4 &= e_2(s)\cos 2m\pi y + e_4(s), \\ e_2(s) &= \frac{(m\pi)^4}{8}\left[\theta_0^2 + \frac{1}{8}\theta_0^4 + 2\theta_0^2 \operatorname{cn}^2(\omega_1 s, k) + \frac{1}{8}\theta_0^4 \operatorname{cn}^4(\omega_1 s, k)\right], \\ e_4(s) &= \frac{3}{64}(m\pi\theta_0)^4 \operatorname{cn}^4(\omega_1 s, k) + m\pi\theta_0 \operatorname{cn}(\omega_1 s, k)\int_0^1 \theta_3 \cos m\pi y\, dy, \end{aligned} \tag{27}$$

where $e_4(s)$ is obtained after the use of the second Eq. (19). The second Eq. (16) thus becomes

$$\theta_{3,ss} + \theta_{3,yyyy} - \frac{1}{4}(m\pi\theta_0)^2 \operatorname{cn}^2(\omega_1 s, k)\theta_{3,yy} + (m\pi)^4\theta_0^2 \operatorname{cn}^2(\omega_1 s, k).$$

$$\begin{aligned} &\cos m\pi y \int_0^1 \theta_3 \cos m\pi y\, dy = R_1(s)\cos m\pi y + R_2(s)\cos 3m\pi y, \\ &R_1(s) = (m\pi)^7\left[\left(1 - \frac{13}{16}\theta_0^2 - \frac{13}{128}\theta_0^4\right)\theta_0 \operatorname{cn}(\omega_1 s, k) + \frac{15}{8}\theta_0^3 \operatorname{cn}^3(\omega_1 s, k)\right. \\ &\qquad\qquad\left. + \frac{21}{128}\theta_0^5 \operatorname{cn}^5(\omega_1 s, k)\right], \\ &R_2(s) = (m\pi)^7\left[-\frac{5}{16}\theta_0^3 \operatorname{cn}(\omega_1 s, k) - \frac{5}{128}\theta_0^5 \operatorname{cn}(\omega_1 s, k) - \frac{3}{8}\theta_0^3 \operatorname{cn}^3(\omega_1 s, k)\right. \\ &\qquad\qquad\left. - \frac{5}{128}\theta_0^5 \operatorname{cn}^5(\omega_1 s, k)\right]. \end{aligned} \tag{28}$$

The solution of partial differential equation (28) is of the following form:

$$\theta_3 = \theta_3^{(0)} + \theta_3^{(1)} + \theta_3^{(2)}, \tag{29}$$

where $\theta_3^{(0)}$ is the solution of the reduced equation. $\theta_3^{(1)}$ and $\theta_3^{(2)}$ are particular solutions of the differential equations obtained by taking $R_2(s) \equiv 0$ and $R_1(s) \equiv 0$ respectively. An examination of the second Eq. (18) reveals that

$$\theta_3^{(0)} = S_1(\sigma) \cos m\pi y, \qquad \theta_3^{(0)} = S_2(\sigma) \cos r\pi y, \qquad \sigma = \omega_1 s, \qquad r \neq m, \tag{30}$$

where S_1 and S_2 satisfy the following differential equations which are obtained by substituting Eqs. (30) into the reduced equation

$$S_{1,\sigma\sigma} = [2k^2 \operatorname{sn}^2 (\sigma, k) - 1]S_1 , \tag{31}$$

$$S_{2,\sigma\sigma} = [6k^2 \operatorname{sn}^2 (\sigma, k) - (1 + 4k^2)]S_2 . \tag{32}$$

Differential equations (31) and (32) are the Jacobian forms of the generalized Lamé equation. The solutions of these equations are:

$$S_\alpha = B\Lambda_\alpha^{(1)} + C\Lambda_\alpha^{(2)}, \qquad (\alpha = 1, 2) \tag{33}$$

$$\begin{aligned} \Lambda_1^{(1)} &= \operatorname{cn} (\sigma, k), \\ \Lambda_1^{(2)} &= \operatorname{cn} (\sigma, k)[(1 - k^2)\sigma + \operatorname{dn} (\sigma, k) \operatorname{sc} (\sigma, k) - E(\sigma, k)]; \end{aligned} \tag{34}$$

$$\begin{aligned} \Lambda_2^{(1)} &= \prod_{r=1}^{2} \frac{H(\sigma + \sigma_r)}{\Theta(\sigma)} \exp [-\sigma Z(\sigma_r)], \\ \Lambda_2^{(2)} &= \prod_{r=1}^{2} \frac{H(\sigma - \sigma_r)}{\Theta(\sigma)} \exp [\sigma Z(\sigma_r)], \end{aligned} \tag{35}$$

where σ_1 and σ_2 are chosen to satisfy the following two independent equations:

$$\begin{aligned} \operatorname{sn} \sigma_1 \operatorname{cn} \sigma_1 \operatorname{dn} \sigma_1 + \operatorname{sn} \sigma_2 \operatorname{cn} \sigma_2 \operatorname{dn} \sigma_2 &= 0, \\ (\operatorname{cn} \sigma_1 \operatorname{ds} \sigma_1 + \operatorname{cn} \sigma_2 \operatorname{ds} \sigma_2)^2 - \operatorname{ns}^2 \sigma_1 - \operatorname{ns}^2 \sigma_2 &= -(1 + 4k^2). \end{aligned} \tag{36}$$

Here $\Theta(u)$, $H(u)$, and $Z(u)$ are Jacobian Theta, Eta and Zeta functions respectively [4]; $E(u)$ is the fundamental elliptic integral of the second kind; sn u, cn u, and dn u are Jacobian elliptic functions and sc u = sn u/cn u, ds u = dn u/sn u, ns u = 1/sn u as originated by Glaisher. These functions are tabulated in [6].

The general solution of an equation similar to (32) was first given by Hermite [5]. Only the first of Eqs. (34) can be extracted from this solution by use of properties of these functions. The second solution $\Lambda_1^{(2)}$ is obtained upon reducing the order of the differential Eq. (31) by letting $\Lambda_1^{(2)} = \Lambda(\sigma) \operatorname{cn} (\sigma, k)$ and upon integrating the resulting equation for $\Lambda(\sigma)$. It can be seen that while $\Lambda_1^{(1)}$ is a doubly-periodic function $\Lambda_1^{(2)}$ is not periodic.

Particular solutions $\theta_3^{(1)}$ and $\theta_3^{(2)}$ of Eqs. (28) are of the following forms:

$$\theta_3^{(1)} = S^{(1)}(\sigma) \cos m\pi y, \qquad \theta_3^{(2)} = S^{(2)}(\sigma) \cos 3m\pi y. \tag{37}$$

Substitution of Eqs. (37) into the first of Eqs. (28) with $R_2(s) \equiv 0$ and $R_1(s) \equiv 0$ respectively, results in

$$S^{(1)}_{,\sigma\sigma} - [2k^2 \operatorname{sn}^2 (\sigma, k) - 1]S^{(1)} = R_1(\sigma), \tag{38}$$

$$S^{(2)}_{,\sigma\sigma} - [6k^2 \operatorname{sn}^2 (\sigma, k) - (1 + 4k^2)]S^{(2)} = R_2(\sigma). \tag{39}$$

Particular solutions of these equations are obtained by Lagrange's method of variation of parameters. Hence*

$$S^{(\alpha)}(\sigma) = \frac{1}{\Delta_\alpha} \int_0^\sigma [\Lambda^{(1)}_\alpha(\sigma) \cdot \Lambda^{(2)}_\alpha(t) - \Lambda^{(2)}_\alpha(\sigma)\Lambda^{(1)}_\alpha(t)]R_\alpha(t)\, dt, \tag{40}$$

$$\Delta_1 = -(1 - k^2), \qquad (\alpha = 1, 2) \tag{41}$$

$$\Delta_2 = 2\, \frac{\prod_{i=1}^{2} H(\sigma_i)H(-\sigma_i)}{\Theta^2(0)} \sum_{i=1}^{2} \operatorname{cn} \sigma_i \operatorname{dn} \sigma_i \operatorname{ns} \sigma_i\, , \tag{42}$$

where σ_1 and σ_2 must be solved from Eq. (36). Finally, the use of the initial conditions given by Eq. (20), namely, $\theta_3 = \theta_{3,s} = 0$ for $s = 0$ reduce the general solution $\theta_3(y, \sigma)$ to:

$$\theta_3(y, \sigma) = S^{(1)}(\sigma) \cos m\pi y + S^{(2)}(\sigma) \cos 3m\pi y. \tag{43}$$

$e_4(s)$ of Eq. (27) thus becomes

$$e_4(s) = \frac{3}{64} (m\pi\theta_0)^4 \operatorname{cn}^4 (\omega_1 s, k) + \frac{1}{2} m\pi\theta_0 S^{(1)}(\omega_1 s) \operatorname{cn} (\omega_1 s, k) \tag{44}$$

This completes the solution of the equations corresponding to λ^4 and λ^5. Further approximation requires more tedious analysis. However, since larger initial deflection will cause extremely high axial tension in the bar, the validity of Hooke's Law must be examined before taking any further steps toward improvement.

There still remains the examination of the question of convergence. This is impossible at this point. Although a term, σ, appears in the expression for $\Lambda_1^{(2)}$, this does not necessarily mean a resonance effect, since further approximations are necessary to examine the series in σ for the question of stability.

An estimate of region of stability of Eqs. (31) and (32) can be made for small k. In this case $\sin \sigma$ can be used in place of $\operatorname{sn} \sigma$. Therefore Eqs. (31) and (32) become Mathieu equations whose stability regions in terms of parameter k^2 are well known [7], and can easily be seen to contain those of the present equations.

4. Vibration produced by an arbitrary initial deflection. Analysis of the preceding section is based on an initial condition corresponding to an initial sinuzoidal deflection of the hinged bar. Differential equations must be re-solved when the initial deflection is arbitrarily prescribed, since these equations are non-linear. The problem of non-linear vibrations of the elastic string following an initial deflection has its analogue in the case of hinged elastic bars. Following a similar method, as that of [3] an integral equation is obtained below which may be solved by the method of successive approximations. This

*Eq. (40) for $\alpha = 1$, can be integrated in closed form, in terms of elliptic functions. This result will not be given here, however.

equation represents the λ^3-approximation of the problem since it is obtained by integrating Eq. (22). Let

$$\theta_1(y, s) = \sum_m A_m S_m(s) \cos m\pi y,$$
$$\theta_1(y, 0) = \sum_m A_m \cos m\pi y, \qquad S_m(0) = 1, \tag{45}$$

where A_m are the Fourier coefficients for the initial deflection function $\theta_1(y, 0)$. Substitution of Eq. (45) into Eq. (22) leads to

$$S_{m,ss} + (m\pi)^4 S_m = -\frac{(m\pi)^2}{4} S_m \sum_n A_n^2 S_n^2 . \tag{46}$$

The required result is obtained by integrating Eq. (46):

$$S_m(s) = \cos m^2\pi^2 s - \frac{1}{4}\int_0^s \sin m^2\pi^2(s - t) S_m(t) \sum_n A_n^2 S_n^2(t)\, dt. \tag{47}$$

There remains the problem of solving ϵ_4, ϵ_6, $\cdots$; θ_3, θ_5, etc., for an arbitrary initial deflection. However, corresponding differential equations given by (15) and (16) are linear. Therefore superposition is valid after $\theta_1(y, s)$ is solved from non-linear integral Eq. (47).

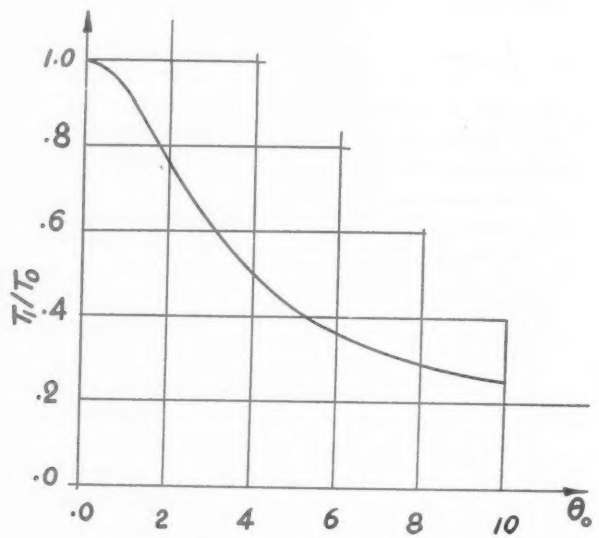

FIG. 2. Period versus initial deflection.

On Fig. 2 the ratio of non-linear period over linear period, T_1/T_0, is plotted against initial deflection angle θ_0. It is seen that classical theory is correct only for vanishing θ_0. Fig. (3) represents the axial stress multiplied by a scale $(1/m^2\pi^2 E)$ against T_1/T_0.

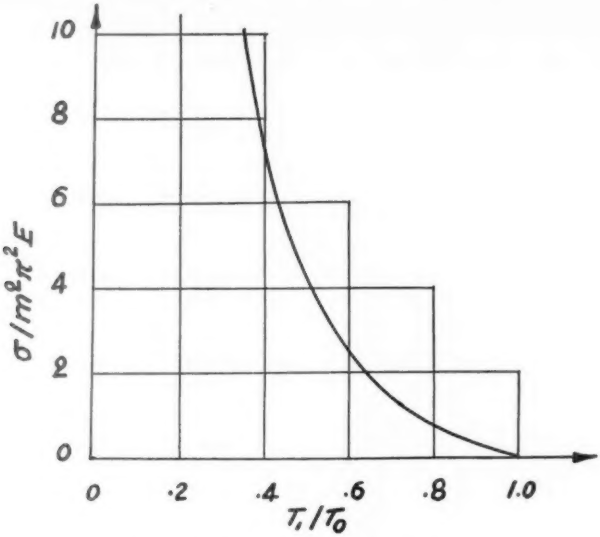

FIG. 3. Axial stress versus period.

It is seen that axial stress increases very rapidly with the decrease of T_1/T_0 or with the increase of frequency ratios.

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ENERGY THEOREMS AND CRITICAL LOAD APPROXIMATIONS IN THE GENERAL THEORY OF ELASTIC STABILITY*

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1. Introduction. When the ordinary uniform pinned-end column buckles under a critical load $P_1 = \pi^2 EI/l^2$, the potential energy measured from the straight compressed form is

$$V = \frac{1}{2} EI \int_0^l y''^2 \, dx - \frac{1}{2} P_1 \int_0^l y'^2 \, dx$$

or

$$\frac{2V}{EI} = \int_0^l y''^2 \, dx - \frac{\pi^2}{l^2} \int_0^l y'^2 \, dx \tag{1}$$

and is zero since $Y \propto \sin \pi x/l$.

Let y now be a deflection of any form which satisfies the end conditions. Then V

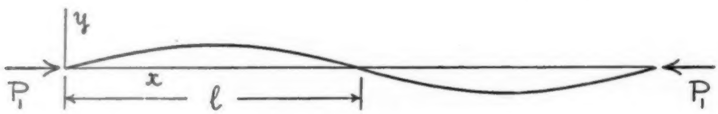

FIG. 1.

as given by (1) is *positive* unless $y \propto \sin \pi x/l$, in which case it is zero. This follows readily from a Fourier expansion[1] of y, or from Wirtinger's inequality,[2] which is that

$$\int_0^{2\pi} z^2 \, dx < \int_0^{2\pi} z'^2 \, dx, \tag{2}$$

unless $z = A \cos x + B \sin x$, provided that

$$z(0) = z(2\pi) \qquad \text{and} \qquad \int_0^{2\pi} z \, dx = 0.$$

For if we reflect the bar with its deflection y in its right hand end ($x = l$) and combine the inverted reflection with the original bar (Fig. 1), we have a bar of length $2l$, with $y'(0) = y'(2l)$, and from (1)

$$\frac{4V}{EI} = \int_0^{2l} y''^2 \, dx - \frac{\pi^2}{l^2} \int_0^{2l} y'^2 \, dx.$$

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*Received February 5, 1951.

[1]See for instance R. V. Southwell, *An introduction to the theory of elasticity for engineers and physicists*, Oxford University Press, 1941, p. 444.

[2]G. H. Hardy, E. H. Littlewood, G. Polya, *Inequalities*, Cambridge University Press, 1934, p. 185. The authors are indebted to Prof. Polya for the suggestion that this inequality might serve the purpose.

The substitution $\xi = \pi x/l$ converts this to

$$\frac{4V}{EI} = \left(\frac{\pi}{l}\right)^3\left[\int_0^{2\pi} y''^2\,d\xi - \int_0^{2\pi} y'^2\,d\xi\right], \tag{3}$$

where the prime now means differentiation with respect to ξ, and if we identify y' with z, ξ with x in (2), (2) establishes that V is positive unless y is sinusoidal.

Two conclusions follow. *First*, that since V is the potential energy of the bar under the true critical load P_1, the sinusoidal buckled form is itself stable with respect to disturbances (impulses) which project it into non-sinusoidal forms. It is neutral with respect to sinusoidal disturbances. In the disturbances P_1 remains unchanged. *Second*, that an approximation P_a calculated by inserting an assumed approximate deflection y_a for y in the relation

$$\frac{1}{2}EI\int_0^l y''^2\,dx - \frac{1}{2}P\int_0^l y'^2\,dx = 0 \tag{4}$$

(valid when P is P_1 and y is sinusoidal) will be higher than P_1. For the use of (4) in this way will yield

$$P_a = \frac{EI\displaystyle\int_0^l y_a''^2\,dx}{\displaystyle\int_0^l y_a'^2\,dx}. \tag{5}$$

But since y_a is not sinusoidal we have $V > 0$ and (1) yields

$$P_1 < \frac{EI\displaystyle\int_0^l y_a''^2\,dx}{\displaystyle\int_0^l y_a'^2\,dx}. \tag{6}$$

Here we have an inequality (2) available from pure mathematics, and can use it to establish either the first or the second conclusion.

In a plate problem the corresponding inequality, establishing that the potential energy, (strain energy of bending minus work of critical loads on buckling displacements), is positive for any displacement differing from the true buckling displacement, is a much more elaborate one, although it can of course be formulated. A proof by means of Fourier series is feasible for the simpler cases, such as the rectangular plate with four simply-supported edges, but a proof for the more difficult cases such as four clamped edges is hardly to be expected. Still less can we hope to obtain such a proof for more complex systems such as shells, or combinations of structural elements such as stiffened plates and shells, or the general problem of elastic stability with respect to infinitesimal displacements.

But if we are *given* that the buckled form is itself not unstable, this datum establishes the inequality, and we can then use it to prove that the energy approximations to the critical loads will be too high. In the remainder of the paper we do this for the general stability problem. If the buckled state *were* itself unstable, the energy approximation to the critical load would be too low. Thus the usual assumption in practical calculations that the approximation will be too high is equivalent to the assumption that in the idealized version of the problem there *is* a buckled state which is itself not unstable.

2. Formulation of the general equations. An arbitrary elastic solid has initial stress specified by the usual Cartesian components S_{ij} $(i, j = 1, 2, 3)$, which maintain equilibrium with initial body force F_i per unit volume and surface force T_i per unit area on a surface element whose outward direction cosines are ν_i. We have then the differential equations of equilibrium (with the summation convention for repeated indices, and subscripts after a comma indicating differentiation with respect to the corresponding co-ordinates)

$$S_{ij,i} + F_i = 0 \tag{7}$$

and the boundary conditions of equilibrium

$$S_{ij}\nu_i = T_i\,. \tag{8}$$

The stress S_{ij} is not necessarily entirely due to F_i and T_i. It may be initial or thermal stress existing in the absence of F_i and T_i.

This state of stress will be referred to as state I, and x_i are the co-ordinates of material points in this state (not in the unstressed state). For the present, we suppose that it is stable. A second state, state II, is derived from it by the application of additional body force ΔF_i and additional surface force ΔT_i. The displacement caused is expressed by Cartesian components u_i (not Δu_i), and it is affected by the presence of the initial stress. The stress in state II is of course different from S_{ij}. To specify it we use Trefftz's stress components[3] k_{ij} (in Kappus' notation). These are non-orthogonal. A rectangular block element in state I becomes an elementary parallepiped in state II, and these stress components refer to the directions of its edges. The advantage of using them is that they lead to relatively simple equations. We may write

$$k_{ij} = S_{ij} + \tau_{ij} \tag{9}$$

and $\tau_{ij} = \tau_{ji}$ since both $S_{ij} = S_{ji}$ and $k_{ij} = k_{ji}$. Even where the τ_{ij} vanish, this state of stress need not be identical with that expressed by the S_{ij} of state I, on account of the different specification of stress components.

The differential equations of equilibrium[4] satisfied by τ_{ij} are

$$\tau_{ij,i} + (S_{jk}u_{i,k}), j + \Delta F_i = 0 \tag{10}$$

after neglecting "non-linear" terms $(\tau_{jk}u_{i,k})_{,j}$, and so restricting the investigation to τ_{ij} small compared with S_{ij}—more precisely to the largest τ_{ij} small compared with the largest S_{ij}.

The boundary conditions of equilibrium satisfied by τ_{ij} are

$$\tau_{ij}\nu_j + S_{jk}u_{i,k}\nu_j = \Delta T_i\,. \tag{11}$$

When $S_{ij} = 0$, (10) and (11) reduce as they should to the equations of the ordinary theory of elasticity.

Equations (10) and (11) do not involve any stress-strain relations. Being concerned with small departures from state I, we assume that small strain components

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{12}$$

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[3] E. Trefftz, *Zur Theorie der Stabilität des elastischen Gleichgewichts*, Z. angew. Math. Mech. 12, 160 (1933).

[4] See the reference in footnote (3).

are related to the small stress components τ_{ij} by the usual form of Hooke's Law. It is convenient to take this in the general form appropriate to the homogeneous anisotropic solid, and to express this form in a changed notation. Write $\tau_{11} = \tau_1$, $\tau_{22} = \tau_2$, $\tau_{33} = \tau_3$, $\tau_{12} = \tau_4$, $\tau_{23} = \tau_5$, $\tau_{31} = \tau_6$ and similarly for the strain components. Then the stress-strain relations are[5]

$$\tau_i = c_{ij}e_j \text{ , where } i \text{ and } j \text{ have the range 1 to 6, and } c_{ij} = c_{ji} \tag{13}$$

The variational principles (stationary potential energy for equilibrium, Castigliano's Theorem, etc.) of the ordinary theory of elasticity can be derived by considering the variation of the strain energy of the body, and using the equations of equilibrium.[6] We follow this method now for the transition from state I to state II, with some modification, using the equations of this article. We thus regard equations of equilibrium (or motion) as basic, and energy principles such as stationary potential energy as derived, rather than vice-versa.

3. A variational principle. From the quantities $e_i (i = 1 \text{ to } 6)$ as functions of the co-ordinates $x_i (i = 1 \text{ to } 3)$ in state I we may form by integration over all the volume elements $d\omega$ of state I the integral

$$U(e) = \frac{1}{2} \int c_{ij}e_ie_j \, d\omega \tag{14}$$

which would be the strain energy in the absence of initial stress. Trefftz[7] has shown that the strain energy acquired in the passage from state I to state II is

$$U(e) + \int S_{ij}e_{ij} \, d\omega + \frac{1}{2} \int S_{jk}u_{i,j}u_{i,k} \, d\omega, \qquad (i, j = 1, 2, 3). \tag{15}$$

We consider the variation of $U(e)$ alone. Let arbitrary variations δu_i be added to the displacements u_i. Then, writing δU for the complete variation of U we have from (14)

$$U(e) + \delta U = \frac{1}{2} \int c_{ij}(e_i + \delta e_i)(e_j + \delta e_j) \qquad (i, j = 1 \text{ to } 6)$$

$$= U(e) + \frac{1}{2} \int c_{ij}(e_i \delta e_j + e_j \delta e_i) \, d\omega + U(\delta e),$$

where

$$U(\delta e) = \frac{1}{2} \int c_{ij} \delta e_i \delta e_j \, d\omega. \tag{16}$$

Since $c_{ij} = c_{ji}$, we have

$$\delta U = \int c_{ij}e_i \delta e_j \, d\omega + U(\delta e) = \int \tau_j \delta e_j \, d\omega + U(\delta e). \tag{17}$$

We now return to the range 1, 2, 3 for i, j and k, and write instead of (17):

$$\delta U = \int \tau_{ij} \, \delta e_{ij} \, d\omega + U(\delta e). \tag{18}$$

[5] A. E. H. Love, *Mathematical theory of elasticity*, Cambridge University Press, 1927, Ch. 3.
[6] E. Trefftz, *Handbuch der Physik*, Vol. 6, Springer, Berlin, 1928, p. 68.
[7] See the reference in footnote (3).

For the integral in (18) we have, taking δe_{ij} from (12),

$$\frac{1}{2}\int \tau_{ij}(\delta u_{i,j} + \delta u_{j,i})\, d\omega = \int \tau_{ij}\, \delta u_{i,j}\, d\omega \qquad (\text{since } \tau_{ij} = \tau_{ji})$$

$$= \int (\tau_{ij}\, \delta u_i)_{,j}\, d\omega - \int \tau_{ij,j}\, \delta u_i\, d\omega.$$

By an application of the divergence theorem to the first of these two integrals we may write the result as

$$\int_{\Sigma} \tau_{ij}\, \delta u_i\, \nu_j\, d\sigma - \int \tau_{ij,j}\, \delta u_i\, d\omega,$$

where $d\sigma$ and Σ refer to the boundary surface and ν_j are the direction cosines of the normal, all in state I. We now eliminate τ_{ij} by use of the boundary conditions (11) in the first integral and the equilibrium equations (10) in the second. The result is

$$\int_{\Sigma} [-S_{jk}u_{i,k}\nu_j + \Delta T_i]\, \delta u_i\, d\sigma + \int [(S_{jk}u_{i,k})_{,j} + \Delta F_i]\, \delta u_i\, d\omega \qquad (19)$$

The first part of the second integral is transformed as follows

$$\int (S_{jk}u_{i,k})_{,j}\, \delta u_i\, d\omega = \int (S_{jk}u_{i,k}\, \delta u_i)_{,j}\, d\omega - \int S_{jk}u_{i,j}\, \delta u_{i,k}\, d\omega$$

$$= \int_{\Sigma} S_{jk}u_{i,k}\, \delta u_i\, \nu_j\, d\sigma - \int S_{jk}u_{i,j}\, \delta u_{i,k}\, d\omega.$$

With this (19) is simplified by cancellation of the two surface integrals involving S_{jk}. Recalling that (19) is equivalent to the integral in (18), we have as a new version of the latter equation

$$\delta U = \int_{\Sigma} \Delta T_i\, \delta u_i\, d\sigma + \int \Delta F_i\, \delta u_i\, d\omega - \int S_{jk}u_{i,j}\, \delta u_{i,k}\, d\omega + U(\delta e). \qquad (20)$$

In this the u_i are the actual displacements caused by the application of the additional forces ΔT_i and ΔF_i, corresponding to the passage from state I to state II, and the δu_i are arbitrary additional displacements. Both the u_i and the δu_i are restricted to smallness by (12) and (13).

Now let S_{ij}, F_i, T_i, ΔF_i, ΔT_i be fixed, but let u_i for the moment be three independent functions of the x_i, not required as yet to be the correct displacements in the passage from state I to state II. The result (20) suggests consideration of a function of these u_i in the form

$$V = U(e) - \int_{\Sigma} \Delta T_i u_i\, d\sigma - \int \Delta F_i u_i\, d\omega + \frac{1}{2}\int S_{jk}u_{i,j}u_{i,k}\, d\omega. \qquad (21)$$

On varying the u_i (as they appear explicitly, and also in $U(e)$) we have

$$\delta V = \delta U - \int_{\Sigma} \Delta T_i\, \delta u_i\, d\sigma - \int \Delta F_i\, \delta u_i\, d\omega + \frac{1}{2}\int S_{jk}\, \delta(u_{i,j}u_{i,k})\, d\omega. \qquad (22)$$

Now let u_i be given the actual values of state II. Then on account of (20) we have

$$\delta V = \int S_{ik}\left[\frac{1}{2}\,\delta(u_{i,j}u_{i,k}) - u_{i,j}\,\delta u_{i,k}\right]d\omega + U(\delta e). \tag{23}$$

But

$$\tfrac{1}{2}S_{ik}\,\delta(u_{i,j}u_{i,k}) = \tfrac{1}{2}S_{ik}[(u_i + \delta u_i)_{,j}(u_i + \delta u_i)_{,k} - u_{i,j}u_{i,k}]$$

$$= \tfrac{1}{2}S_{ik}[\delta u_{i,j}\,u_{i,k} + u_{i,j}\,\delta u_{i,k} + \delta u_{i,j}\,\delta u_{i,k}].$$

Since $S_{jk} = S_{kj}$ the first term in the brackets can be combined with the second to give the result

$$\tfrac{1}{2}S_{ik}[2u_{i,j}\,\delta u_{i,k} + \delta u_{i,j}\,\delta u_{i,k}].$$

Then (23) reduces to

$$\delta V = 0 + U(\delta e) + \frac{1}{2}\int S_{jk}\,\delta u_{i,j}\,\delta u_{i,k}\,d\omega, \tag{24}$$

the zero indicating that the first order (in δu_i) variation of V vanishes. This property of course would be characteristic of the potential energy in state II as an equilibrium state. Referred to state I as zero, the potential energy consists of the strain energy (15) together with the potential energy of the body and surface forces, which is given for state II by

$$-\int_{\Sigma}(T_i + \Delta T_i)u_i\,d\sigma - \int (F_i + \Delta F_i)u_i\,d\omega.$$

It can be shown by means of (7) and (8) that the terms here in T_i and F_i cancel the middle term in (15), and hence that V as given by (21) is in fact the potential energy of state II when the u_i in (21) denote the actual displacements of state II.

4. The stability of state II. Our object being to deduce a generalization of the inequality (6) when the stability of state II is given, we now seek a necessary condition for this stability.

Let the particles of the body be projected from state II by some disturbance. Then at time t they are in motion with displacements δu_i (functions of the co-ordinates x_i of the particles in state I), and corresponding to δu_i we have additions $\delta\tau_{ij}$ to the Trefftz stress components (9). There is, by hypothesis, no change in the forces $F_i + \Delta F_i$, $T_i + \Delta T_i$ of state II except at fixed supports where reactions may be induced. During the motion these are carried with the particles on which they act in state II. The equations of motion are, from (10)

$$(\tau_{ij} + \delta\tau_{ij})_{,j} + [S_{jk}(u_i + \delta u_i)_{,k}]_{,j} + \Delta F_i = \rho\delta\ddot{u}_i\,,$$

where $\delta\ddot{u}_i = \partial^2\delta u_i/\partial t^2$ (the acceleration) and ρ is the density. Subtracting (10) we have

$$\tau_{ij,j} + (S_{jk}\delta u_{i,k})_{,j} = \rho\delta\ddot{u}_i\,.$$

Multiplying by $\delta\dot{u}_i$ and integrating over the volume we find

$$\int \tau_{ij,j}\,\delta\dot{u}_i\,d\omega + \int (S_{jk}\,\delta u_{i,k})_{,j}\,\delta\dot{u}_i\,d\omega = \frac{d}{dt}\int \frac{1}{2}\,\rho\,\delta\dot{u}_i\,\delta\dot{u}_i\,d\omega, \tag{25}$$

and the term on the right is the time derivative of the kinetic energy. The first integral on the left of (25) can be written as

$$\int (\delta\tau_{ij}\ \delta\dot{u}_i)_{,j}\ d\omega - \int \delta\tau_{ij}\ \delta\dot{u}_{i,j}\ d\omega$$

and after transformation of the first of these integrals by the divergence theorem, as

$$\int_\Sigma \delta\tau_{ij}\ \delta\dot{u}_i\ \nu_j\ d\sigma - \int \delta\tau_{ij}\ \delta\dot{u}_{i,j}\ d\omega.$$

Similarly the second integral on the left of (25) transforms into

$$\int_\Sigma S_{jk}\ \delta u_{i,k}\ \delta\dot{u}_i\ \nu_j\ d\sigma - \int S_{jk}\ \delta u_{i,k}\ \delta\dot{u}_{i,j}\ d\omega.$$

Introducing these transformations in (25), and writing T for the kinetic energy, we have, with some rearrangement,

$$\frac{dT}{dt} + \int (\delta\tau_{ij} + S_{jk}\ \delta u_{i,k})\ \delta\dot{u}_{i,j}\ d\omega = \int_\Sigma (\delta\tau_{ij} + S_{jk}\ \delta u_{i,k})\ \delta\dot{u}_i\ \nu_j\ d\sigma. \tag{26}$$

The bracket appearing in the integral on the right is, by (11), the addition to ΔT_i accompanying the motion, and this is zero by hypothesis except at fixed supports, where the $\delta\dot{u}_i$ vanish. The integral therefore vanishes. The integral on the left of (26) is the same as

$$\frac{d}{dt}\left\{\frac{1}{2}\int \delta\tau_{ij}\ \delta e_{ij}\ d\omega + \frac{1}{2}\int S_{jk}\ \delta u_{i,j}\ \delta u_{i,k}\ d\omega\right\}. \tag{27}$$

This is readily verified by changing $\delta\tau_{ij}\delta e_{ij}$ to $\delta\tau_i\delta e_i (i = 1 \text{ to } 6)$, then to $c_{ij}\delta e_i\delta e_j$, carrying out the differentiation with respect to t, and making the combinations of terms permitted by $c_{ij} = c_{ji}$ and $S_{jk} = S_{kj}$. It is evident that the bracket in (27) is identical with δV as given by (24). We can therefore re-express (26) as

$$\frac{dT}{dt} + \frac{d\ \delta V}{dt} = 0 \tag{28}$$

showing that $T + \delta V$ remains constant during the motion following the projection from the equilibrium state II. This of course is the energy equation of this motion, and exhibits δV as the potential energy referred to state II under the conditions of this motion—no change of body and surface force except at fixed supports.

Stability of state II implies an immediate decrease in the kinetic energy following the projection from state II, and therefore an immediate increase of δV. Thus stability means that δV as given by (24) is positive for arbitrary δu_i. If it is given that state II is not unstable, (24) is not negative.

We may take the u_i to be zero, as a special case, state II then being the same as state I. Evidently the equilibrium in state I, under the initial stress S_{ij}, will be unstable when the right hand side of (24) is negative for any δu_i which vanish at fixed supports.

The value which V, as given by (21), takes when the u_i are the correct displacements

of state II can be reduced to a simpler and useful form. Writing $-I$ for the last integral in (21) we have

$$-I = \frac{1}{2}\int S_{ik}u_{i,j}u_{i,k}\,d\omega$$

$$= \frac{1}{2}\int (S_{ik}u_{i,k}u_i)_{,j}\,d\omega - \frac{1}{2}\int (S_{ik}u_{i,k})_{,j}u_i\,d\omega$$

$$= \frac{1}{2}\int_\Sigma S_{ik}u_{i,k}u_i\nu_j\,d\sigma - \frac{1}{2}\int (S_{ik}u_{i,k})_{,j}u_i\,d\omega.$$

Using (11) and (10) respectively in the first and second of these integrals we find

$$-I = \frac{1}{2}\int_\Sigma \Delta T_i u_i\,d\sigma - \frac{1}{2}\int_\Sigma \tau_{ij}u_i\nu_j\,d\sigma + \frac{1}{2}\int \Delta F_i u_i\,d\omega + \frac{1}{2}\int \tau_{ij,j}u_i\,d\omega \tag{29}$$

If in the last integral we write

$$\tau_{ij,j}u_i = (\tau_{ij}u_i)_{,j} - \tau_{ij}u_{i,j}\,,$$

the first of the two resulting integrals will, by the divergence theorem, cancel the second integral on the right of (29). Then, observing that, since $\tau_{ij} = \tau_{ji}$ and e_{ij} is given by (12),

$$\frac{1}{2}\int \tau_{ij}u_{i,j}\,d\omega = \frac{1}{2}\int \tau_{ij}e_{ij}\,d\omega = U(e),$$

we can rewrite (29) as

$$-I = \frac{1}{2}\int_\Sigma \Delta T_i\,u_i\,d\sigma + \frac{1}{2}\int \Delta F_i\,u_i\,d\omega - U(e).$$

This form is valid only when u_i are the correct displacements of state II, because (10) and (11) have been incorporated. Returning to (21) we have the corresponding value of V as

$$V = -\frac{1}{2}\int_\Sigma \Delta T_i\,u_i\,d\sigma - \frac{1}{2}\int \Delta F_i\,u_i\,d\omega \tag{30}$$

5. Elastic buckling. We have so far been concerned with two neighboring equilibrium states, state I and state II, the passage from state I to state II being effected by additional surface and body forces ΔT_i , ΔF_i . When the passage is a buckling deformation, there will be no change in body force (e.g. gravity), but for very exceptional problems, and we may take $\Delta F_i = 0$. The surface forces may be taken to change at supports (e.g. the transverse reactions induced when buckling occurs in a column with one end clamped, the other pinned), but not elsewhere (the loads remain unchanged during buckling, moving with the particles they act on).

Let the supports be such that no work is done by the reactions on the buckling displacements (as is true for the common boundary conditions of bars and plates. If work *is* done, as by elastic restraining moments, the elastic restraints may be included in the structure, and *their* fixed supports are then of the assumed type).

Then each integral in (30) vanishes, and therefore $V = 0$. Thus (21) now yields

$$V = U(e) + \frac{1}{2}\int S_{ik}u_{i,j}u_{i,k}\,d\omega = 0 \tag{31}$$

which is a generalized energy relation valid when S_{ik} is a critical state of stress, and the u_i are actual buckling displacements.

But $U(e)$ is necessarily positive for any e_{ij} since it has the form of the strain energy in the absence of initial stress. We have therefore from (31)

$$-I = \frac{1}{2}\int S_{ik}u_{i,j}u_{i,k}\,d\omega < 0 \tag{32}$$

and hence for a critical state I is positive.

The initial stress S_{ik}, now of course a critical state of stress under which buckling from state I to state II is possible, can be represented as

$$S_{ik} = \Gamma S^0_{ik}\,, \tag{33}$$

where S^0_{ik} is a non-critical stress tensor having the same distribution but a non-critical magnitude, and Γ is a positive multiplier. The S^0_{ik} being chosen, we inquire what value of Γ corresponds to a critical state. It now follows from (32) that

$$I^0 = -\frac{1}{2}\int S^0_{ik}u_{i,j}u_{i,k}\,d\omega > 0 \tag{34}$$

which defines I^0.

Introducing (33) in (31), and using the equality in (34) we have

$$\Gamma = \frac{U(e)}{I^0}. \tag{35}$$

This holds, giving the critical value of Γ, when $U(e)$ and I^0 are evaluated from the correct buckling displacements u_i.

We now write Γ' for the quantity which is calculated from the formula (35) using functions u_i' other than u_i in $U(e)$ and I^0, leaving S^0_{ik} in the latter unchanged. We can then inquire what choice of u_i' will yield the least value of Γ'. Let $u_i' = u_i + \delta u_i$, u_i being the correct buckling displacements, and correspondingly write $\Gamma' = \Gamma + \delta\Gamma$. Then

$$\Gamma + \delta\Gamma = \frac{U(e) + \delta U}{I^0 + \delta I^0} = \frac{U(e) + \Gamma\delta I^0 + (\delta U - \Gamma\delta I^0)}{I^0 + \delta I^0}. \tag{36}$$

Let the variations δu_i satisfy the same boundary conditions as the u_i. Then

$$\int_\Sigma \Delta T_i\,\delta u_i\,d\sigma = 0,$$

and we have from (22)

$$\begin{aligned}\delta V &= \delta U + \frac{1}{2}\int S_{ik}\,\delta(u_{i,j}u_{i,k})\,d\omega\\ &= \delta U + \frac{1}{2}\Gamma\int S^0_{ik}\,\delta(u_{i,j}u_{i,k})\,d\omega\\ &= \delta U - \Gamma\,\delta I^0\end{aligned}$$

by the definition of I^0 in (34). Putting this in the last member of (36), and at the same time replacing $U(e)$ by ΓI^0 from (35), we find

$$\delta\Gamma = \frac{\delta V}{I^0 + \delta I^0}. \tag{37}$$

In section 4 we found that when state II is itself stable δV as given by (24) is positive. We now define state II as a buckled state which is itself stable with respect to disturbances which project it into a different configuration (the $u_i + \delta u_i$ not merely proportional to the u_i). These disturbances correspond to the "non-sinusoidal disturbances" of the column in section 1. Our "incorrect" displacements $u_i + \delta u_i$ are of this character, and therefore δV is positive. Since, by (24), δV is zero in the first order (in δu_i) and positive in the second, we have from (37) that $\delta\Gamma$ is also zero in the first order, positive in the second order, and hence that the correct value Γ is the lower bound of the approximations Γ'.

Unlike δV, $\delta\Gamma$ does not terminate with the terms of the second order. It is conceivable that the δu_i could be chosen (not small compared with the u_i) so that the denominator in (37) becomes negative. Then the approximation Γ' would be *smaller* than Γ.

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EXTENDED LIMIT DESIGN THEOREMS FOR CONTINUOUS MEDIA[1]

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Summary. Earlier results [1,2][2] on safe loads for a Prandtl-Reuss material subject to surface tractions or displacements which increase in ratio are here extended to any perfectly plastic material and any history of loading.

1. Introduction. The general limit design problem is concerned with a body or assemblage of bodies made of *perfectly plastic* (i.e. non-workhardening) material and subject to an arbitrary history of loading. In many cases, only the extreme values of the loads are given, but the order in which these loads are applied to the body is not specified. An important question is whether the body will "collapse", that is deform appreciably under essentially constant loads, or whether its deformation will be contained although substantial portions may go plastic. In engineering design, the actual problem is to insure a reasonable margin of safety against such collapse.

In the present paper a somewhat restricted form of this general problem is discussed: the actual history of loading is assumed to be completely specified rather than only the extreme values of all loads. The given loading history may be very simple; for instance, all loads may increase so as to preserve their ratios (*proportional loading*). On the other hand, the loading program may be very elaborate; additional loads may be superimposed on a state of initial stress, as is the case when traffic loads come on a bridge, or when torsion and bending are applied to a shot peened and hence prestressed axle.

The boundary conditions are assumed to be of the *stress type* for a single body or assemblage of bodies. At each point of the surface of the body or assemblage of bodies each component of the surface traction is specified except when the corresponding component of the displacement is prescribed to be zero.

Several terms must be introduced, and a number of concepts must be discussed before the main theorems can be stated and proved.

2. Perfect plasticity. In the following discussion, the general stress-strain relation for a perfectly plastic material will be used so that the results will apply to a wide variety of materials. By definition, a perfectly plastic material in simple tension has a stress-strain diagram of the form shown in Fig. 1. The essential feature of this diagram is the flat yield which produces a sharp boundary between elastic behavior and unrestricted plastic flow at points such as B in Fig. 1.

To describe this kind of behavior mathematically for more general types of loading, it is convenient to use tensor notation. Latin subscripts take the range of values 1, 2, 3,

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[2]Numbers in square brackets refer to the Bibliography at the end of the paper.

and the summation convention concerning repeated letter subscripts is adopted. The coordinates x_i used in the following are rectangular and Cartesian. Differentiation with respect to a coordinate is indicated by a comma followed by the appropriate subscript $(u_{i,j} = \partial u_i/\partial x_j)$.

The mechanical behavior of a perfectly plastic material is completely characterized by its *yield function*. For a homogeneous material, the yield function f depends only on the nine stress components σ_{ij} ; it is positive definite and is symmetric with respect to the conjugate shearing stresses σ_{ij} and σ_{ji} $(i \neq j)$ which are formally treated as independent variables. Plastic flow can occur only under states of stress for which $f = 1$.

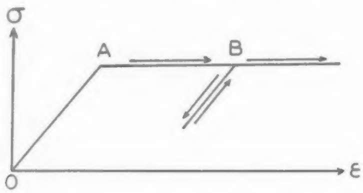

FIG. 1

States of stress for which $f > 1$ are not possible in a perfectly plastic material. For the completely stress-free state $f = 0$; for any other state of stress within the elastic range $0 < f < 1$. In the following, states of stress for which $f < 1$ will be called *safe*.

The most frequently used form of the yield function is $f = s_{ij}s_{ij}/2k^2$, where $s_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}$ is the stress deviation and k the yield stress in simple shear. However, more complicated forms may be used to represent, for instance, various types of anisotropy. For a non-homogeneous material, the yield function may vary from particle to particle.

The stress-strain law of a perfectly plastic material does not contain the strain itself but only the strain rate. Since viscosity effects are disregarded, this law must contain the rates of stress and strain in a homogeneous manner. The strain rate ϵ_{ij} can be decomposed into an elastic component ϵ_{ij}^e and a plastic component ϵ_{ij}^p :

$$\epsilon_{ij} = \epsilon_{ij}^e + \epsilon_{ij}^p . \tag{1}$$

The elastic strain rate ϵ_{ij}^e is related to the rate of stress σ'_{ij} by the generalized form of Hooke's law. For the following, it is not necessary to write down this relation between ϵ_{ij}^e and σ'_{ij} ; it suffices to note that

$$\epsilon_{ij}^e\sigma'_{ij} > 0 \qquad \text{except when} \qquad \epsilon_{ij}^e = \sigma'_{ij} = 0. \tag{2}$$

The plastic strain rate is related to the state of stress. Since viscosity effects are neglected, the stress components must be homogeneous of the order zero in the components of the plastic strain rate. In other terms, the stress tensor σ_{ij} determines the tensor of the plastic strain rate ϵ_{ij}^p only to within an arbitrary factor.

The following geometric representation of states of stress and plastic strain rate is often useful. In a nine-dimensional Euclidean space, consider a *fixed* system of rectangular Cartesian coordinates. The state of stress σ_{ij} will be represented by the point with coordinates proportional to σ_{ij} . The plastic strain rate ϵ_{ij}^p will be represented by the ray with direction cosines proportional to ϵ_{ij}^p . This manner of representing the plastic strain rate by a ray rather than a point is suggested by the fact that the state of stress determines the plastic strain rate only to within an arbitrary factor.

The yield condition $f = 1$ defines a surface in this nine-dimensional space. We assume this *yield surface* to be convex and to possess a unique normal in each of its points. The mechanical significance of this assumption will become clear from the following discussion.

Consider a state of stress for which $f = 1$. As has been shown in earlier papers [3, 4], any plastic strain rate ϵ^p_{ij} associated with this state of stress is represented by the ray which has the direction of the exterior normal of the yield surface at the point σ_{ij}. From this fact and the assumption regarding the yield surface, there follow two important lemmas.

Lemma 1. The stress rate σ'_{ij} and the plastic strain rate ϵ^p_{ij} satisfy

$$\sigma'_{ij}\epsilon^p_{ij} = 0. \tag{3}$$

Proof. If the plastic strain rate is not to vanish, the stress rate must correspond to the change from one point on the yield surface to a neighboring point. Thus, the stress rate is represented by a vector which is tangential to the yield surface, whereas the plastic strain rate is represented by a ray normal to the yield surface. Equation (3) expresses the orthogonality between this vector and ray. The special case $\sigma'_{ij} \equiv 0$ also satisfies Eq. (3).

Lemma 2. There exists a function F, homogeneous of the first order in the components of the plastic strain rate, such that in flow with the plastic strain rate ϵ^p_{ij}, energy is dissipated at the rate $F(\epsilon^p_{ij})$, whereas, for any safe state of stress σ^s_{ij},

$$\sigma^s_{ij}\epsilon^p_{ij} < F(\epsilon^p_{ij}). \tag{4}$$

Proof. If the plastic strain rate ϵ^p_{ij} is associated with the stress σ_{ij}, the rate of dissipation of energy is $\sigma_{ij}\epsilon^p_{ij}$. Since the stress components are homogeneous of the order zero

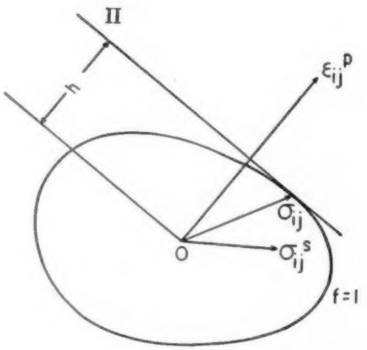

FIG. 2

in the components of the plastic strain rate, the dissipated energy is homogeneous of the first order in these strain rate components. Let Π be the tangent plane of the yield surface at the point σ_{ij} and h its distance from the origin 0 (Fig. 2). The rate of dissipation of energy is then given by $(\epsilon^p_{ij}\epsilon^p_{ij})^{1/2}h$. The inequality (4) expresses the fact that any safe state of stress is represented by a point which lies on the same side of Π

as the origin. (Note that the function F is the supporting function of the convex yield surface.)

3. Collapse. From the point of view of the practical engineer, the term collapse implies that appreciable changes in the geometry of a structure will occur under essentially constant loads. For the purpose of the following discussion, however, it is more convenient to use the term collapse to refer to conditions for which plastic flow would occur under constant loads *if the accompanying change in the geometry of the structure or body were disregarded.* In the discussion of this type of collapse, the equilibrium conditions can be set up for the undeformed rather than the deformed body. This obviously represents a great simplification because the deformation occurring during collapse is not known beforehand but constitutes one of the unknowns of the collapse problem.

To illustrate the relation between these two definitions of collapse, consider first a hollow sphere of uniform wall thickness under gradually increasing interior pressure. The material at the interior surface reaches the yield limit first, but its plastic deformation is contained by the surrounding shell of still elastic material. As the pressure continues to increase, the boundary between the elastic and plastic regions moves towards the exterior surface; only when it has reached this surface does large plastic flow become possible. Let us now compare the simplified stress analysis which neglects all changes of geometry during the loading process to the complete analysis which takes account of these changes. Even in the elastic range, and also in the subsequent range of contained plastic deformation, the hollow sphere expands and its wall thickness diminishes somewhat. If those effects are taken into account, the pressure for which the entire sphere becomes plastic is found somewhat smaller than when they are neglected. However, the difference between the two pressure values is small because the deformations occurring up to the end of the range of contained plastic deformation are of the order of magnitude of elastic deformations. Continuing the analysis into the subsequent range of unrestricted plastic flow, we find that the simplified analysis predicts flow under constant interior pressure, whereas the complete analysis predicts flow under gradually decreasing pressure. In either case the sphere loses its usefulness as a pressure vessel. As far as this first example is concerned, the two definitions of collapse are therefore in substantial agreement as to the pressure under which collapse sets in.

Next, consider a blunt, rigid wedge which is pressed against the flat surface of a large block of perfectly plastic material. It is obvious that the term "collapse" can not be applied to this problem in the sense attributed to this term by the practical engineer. Indeed, as the wedge is pressed into the plastic material, the area of contact continues to increase; thus, the load on the wedge must be increased steadily if plastic flow is to be maintained. On the other hand, there exists a well-defined "collapse load" under which plastic flow would continue if all changes in geometry could be disregarded. The physical meaning of this "collapse load" is less clear than that of the collapse pressure obtained in the preceding example. Obviously, the collapse load in the case of the wedge is not the load under which the first indentation occurs, because some identation takes place even in the elastic range. It is likely, however, that the collapse load indicates the load intensity at which the permanent indentation begins to increase considerably faster than the elastic identation. Thus, the collapse load in the case of the wedge has the nature of a *conventional* yield limit, whereas the collapse pressure in the case of the hollow sphere represents a *natural* yield limit.

To arrive at a mathematical characterization of collapse, let the velocities be denoted by v_i, the surface tractions by T_i, and the body forces (per unit volume) by F_i. Furthermore, use the prime to indicate rates of change and the superscript c to refer to collapse. From the definition of collapse introduced above it follows then that during collapse

$$\int T_i'^c v_i^c \, dS = 0 \qquad \text{and} \qquad F_i'^c = 0 \qquad \text{for some} \qquad v_i^c \neq 0, \tag{5}$$

where the integration is extended over the surface S which bounds the considered body or assemblage of bodies. Indeed, since collapse is to occur under constant loads, the rates $F_i'^c$ must vanish throughout V; moreover, for those components of the surface traction which are prescribed by the boundary conditions the rates T_i' must vanish, whereas the surface velocities v_i^c corresponding to the remaining components of the surface traction must vanish according to our definition of stress boundary conditions. Thus, the integral in the first Eq. (5) is seen to vanish.

An expression for the rate at which work is done during collapse is useful. If the velocities v_i^c considered as functions of the coordinates are continuous and have continuous first derivatives, the principle of virtual displacements yields the following equation:

$$\int T_i^c v_i^c \, dS + \int F_i^c v_i^c \, dV = \int \sigma_{ij}^c \epsilon_{ij}^c \, dV. \tag{6}$$

4. Admissible states. Consider first a state of stress for which the components σ_{ij} are continuous functions of the coordinates. Such a state is called *statically admissible* if it satisfies (i) the conditions of equilibrium

$$\sigma_{ij,j} + F_i = 0 \tag{7}$$

throughout V and (ii) the boundary condition

$$\sigma_{ij} n_j = T_i \tag{8}$$

on those portions of the surface where the component T_i of the surface traction is given. In (8), the unit vector along the exterior normal of S is denoted by n_i.

The preceding definition may be generalized to include stress fields with a finite number of surfaces of discontinuity. On either side of such a surface, the stresses must then satisfy (7). Moreover, if n_j^* denotes the unit normal vector of the surface of discontinuity, the expression $\sigma_{ij} n_j^*$ must have the same value whether it is evaluated from the stresses on one or the other side of the surface of discontinuity.

A velocity field v_i is called *kinematically admissible* if the velocity component v_i vanishes on those portions of the surface S where the corresponding component T_i of the surface traction is not prescribed. A kinematically admissible velocity field may represent rigid body motions for certain portions of the body and genuine deformations for the remainder. Special discontinuous velocity fields are also permissible and often useful; they will be discussed in some detail later. For the present, however, we consider only kinematically admissible velocity fields for which the velocity components are continuous functions of the coordinates.

Such a velocity field is said to define a *kinematically admissible state of collapse* if the

rate at which the actual surface tractions and body forces do work on the velocities v_i^k equals or exceeds the rate of dissipation of energy computed from the strain rates

$$\epsilon_{ij}^k = \tfrac{1}{2}(v_{i,j}^k + v_{j,i}^k) \tag{9}$$

treated as purely plastic strain rates. Thus, for a kinematically admissible collapse state

$$\int T_i v_i^k \, dS + \int F_i v_i^k \, dV \geq \int F(\epsilon_{ij}^k) \, dV. \tag{10}$$

5. Collapse theorems. The following theorems were previously established for special cases [1, 2]; they can now be shown to hold generally.

Theorem 1. If all changes in geometry occurring during collapse are neglected, all stresses are found to remain constant during collapse.

Proof. Applying the principle of virtual work to the velocity field and the rates of change of the surface tractions, body forces, stresses and strains during collpase, we obtain

$$\int T_i'^c v^c \, dS + \int F_i'^c v_i^c \, dV = \int \sigma_{ij}'^c \epsilon_{ij}^c \, dV. \tag{11}$$

According to (5), the left-hand side of (11) must vanish for the considered collapse state. The strain rate on the right-hand side of (11) can be decomposed into its elastic and plastic components; thus,

$$\int \sigma_{ij}'^c \epsilon_{ij}^{ec} \, dV + \int \sigma_{ij}'^c \epsilon_{ij}^{pc} \, dV = 0. \tag{12}$$

The second integral in (12) vanishes according to (3). The relation (2) shows then that (12) can be satisfied only if the stress rate $\sigma_{ij}'^c$ vanishes throughout V.

Theorem 2. If a safe statically admissible state of stress can be found at each stage of loading, collapse will not occur under the given loading schedule.

Proof. Suppose this theorem to be false. Then, at some stage of loading, a collapse state v_i^c would exist although a safe statically admissible state of stress σ_{ij}^s could be found. Applying the principle of virtual work to the actual surface tractions T_i^c and body forces F_i^c at this collapse stage, the stresses σ_{ij}^s with which these are in equilibrium, the velocities v_i^c , and the corresponding strain rates ϵ_{ij}^c , we obtain

$$\int T_i^c v_i^c \, dS + \int F_i^c v_i^c \, dV = \int \sigma_{ij}^s \epsilon_{ij}^c \, dV. \tag{13}$$

According to Theorem 1, the stresses and hence the elastic strain remain constant during the collapse described by the field v_i^c . Thus, the strain rate ϵ_{ij}^c is purely plastic. From the first part of Lemma 2 it follows then that the right-hand side of (6) can be written as $\int F(\epsilon_{ij}^c) \, dV$. Combining this form of (6) with (13), we obtain

$$\int \sigma_{ij}^s \epsilon_{ij}^c \, dV = \int F(\epsilon_{ij}^c) \, dV \tag{14}$$

which is in contradiction to the second part of Lemma 2, since the strain rate ϵ^c_{ij} is purely plastic.

Theorem 3. As long as collapse does not occur, a safe statically admissible state of stress can be found at each stage of loading.

To prove this converse of Theorem 2, consider a generic stage of loading defined by the surface tractions T_i and the body forces F_i. If the actual stresses at this stage of loading are denoted by σ_{ij}, we have $f(\sigma_{ij}) \leq 1$, because collapse is not supposed to occur.

Consider now the loading specified by NT_i and NF_i, where N is constant throughout the body. For $N = 1$, collapse does not occur according to the condition of the theorem. Since the equations of equilibrium are linear in the stresses, body forces, and surface tractions, the stresses $N\sigma_{ij}$ will be in equilibrium with NT_i and NF_i. Moreover, it follows from the convexity of the yield surface that $f(N\sigma_{ij}) < 1$ if $N < 1$. Thus, collapse can occur under the loads NT_i, NF_i only if $N > 1$. Let σ^c_{ij} denote the actual stresses for such a state of collapse. These stresses are in equilibrium with NT_i, NF_i and satisfy $f(\sigma^c_{ij}) \leq 1$. Therefore, the stresses σ^c_{ij}/N are in equilibrium with T_i, F_i and satisfy $f(\sigma^c_{ij}/N) < 1$; in other terms, these stresses define a safe statically admissible state of stress for the loads T_i, F_i.

Theorem 4. If a kinematically admissible collapse state can be found at any stage of loading, collapse must impend or have taken place previously.

Proof. Suppose this theorem to be false. According to Theorem 3, a safe state of stress σ^s_{ij} could then be found in spite of the existence of a kinematically admissible collapse state v^k_i. Applying the principle of virtual work to the actual surface tractions T_i and body forces F_i at the considered stage of loading, the stresses σ^s_{ij} with which these are in equilibrium, the velocities v^k_i and the corresponding strain rates ϵ^k_{ij}, we obtain

$$\int T_i v^k_i \, dS + \int F_i v^k_i \, dV = \int \sigma^s_{ij} \epsilon^k_{ij} \, dV. \tag{15}$$

On the other hand, we may use (10) since v^k_i is a kinematically admissible collapse state. Thus,

$$\int \sigma^s_{ij} \epsilon^k_{ij} \, dV \geq \int F(\epsilon^k_{ij}) \, dV \tag{16}$$

which is in contradiction to (4) since the strain rates ϵ^k_{ij} of a kinematically admissible collapse state are treated as purely plastic.

6. Discontinuous velocity fields. It is often useful to consider discontinuous velocity fields. However, it should be kept in mind that in plastic flow, as distinct from fracture, actual discontinuities cannot occur across a fixed surface. The type of discontinuity to be considered in the following is simply an idealization of a continuous distribution in which the velocity changes very rapidly across a thin transition layer (Fig. 3). This idealization is permissible provided the stresses on the assumed discontinuity surface are chosen as the limiting values of the stresses on the surfaces bounding the transition layer as the thickness of this layer approaches zero. For plane plastic flow in a Prandtl-Reuss material, for instance, the line of discontinuity must be a shear line for each of the stress fields on the two sides of the line of discontinuity. If the yield function f

depends upon the mean normal stress, it will be found that a discontinuity in tangential velocity implies separation or overlap of the material on the two sides of the discontinuity. In such a case, the actual transition layer must have appreciable thickness, but the idealization to a discontinuity surface may still be useful.*

The theorems of Sec. 5 are obviously valid in the presence of a transition layer. They will, therefore, remain valid in the limit as the thickness of the transition layer approaches

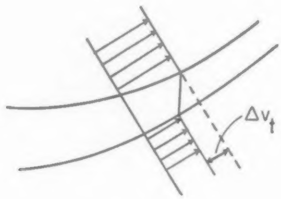

FIG. 3

zero, provided it is kept in mind that the rate of dissipation of energy in the transition layer approaches a finite value in the limit. If the transition layer is replaced by a surface of discontinuity, the expression $\int F(\epsilon_{ij})\, dV$ must, therefore, be replaced by

$$\int F(\epsilon_{ij})\, dV + \int T_i \Delta v_i \, dA \tag{17}$$

where dA is the element of area of the discontinuity surface, T_i the traction and Δv_i the velocity jump across this surface. Thus, in the definition (10) of a kinematically admissible collapse state, the right-hand side must be replaced by (17).

Actual discontinuity of velocity can occur in the case of an assemblage of bodies; for instance slip may occur between a punch and the indented material. If there is no friction between the bodies of an assemblage, Theorems 1 through 4 remain valid in spite of such discontinuities in the velocity because no energy is dissipated on the contact surface in the absence of friction.

7. Additional theorems. The following theorems are intuitively obvious but their statement and indication of proof seems worthwhile.

Theorem 5. Addition to the body of (weightless) material cannot result in a lower collapse load. The proof follows directly from the fact that the collapse stresses σ^c_{ij} for the original body and zero stresses in the added material constitute a statically admissible state for the new body.

Corollary. Removal of material cannot increase the collapse load.

If the yield surface of one material contains that of a second material, the first material will be said to have *higher yield strength* than the second.

Theorem 6. Increasing the yield strength of the material in any region of a perfectly plastic body cannot weaken the body.

The proof follows from the fact that any statically admissible state of stress which is safe for the unimproved body is also safe for the improved body.

*Application to soil mechanics provides an excellent illustration and will be treated in a separate paper.

Theorem 7. Initial stresses or deformations have no effect on collapse providing the geometry is essentially unaltered.

To prove this, we note that an initial or residual state does not affect any of the equations or statements made. This means, for example, that settlement of supports of a continuous structure or initial plastic torsion of a bar subsequently bent or pulled does not affect the limit loads provided the geometry is not changed appreciably.

It is probably best to end on a note of caution. Just as in elasticity, the concept of an essentially unaltered geometry rules out buckling which must, therefore, be studied separately.

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AN INTEGRAL EQUATION APPROACH TO THE PROBLEM OF WAVE PROPAGATION OVER AN IRREGULAR SURFACE*

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In this paper we propose to outline a certain approximate technique for solving the wave equation under the following conditions. Consider a surface S which resembles somewhat the surface drawn in Fig. 1; it stretches out to infinity along a horizontal

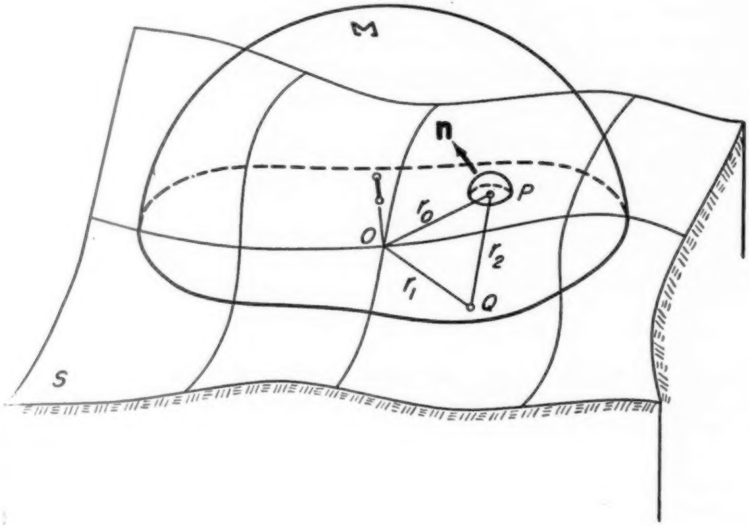

FIG. 1.

plane and is composed of "large" corrugations or bumps—large in the sense that the radius of curvature at any point of S is much larger than a wavelength. Above S there is a field $\psi\ (x, y, z)e^{-i\omega t}$ which varies harmonically in time with the radial frequency ω and which satisfies the wave equation

$$\nabla^2\psi + k^2\psi = -4\pi\tau, \tag{1}$$

k being the propagation constant ω/c and τ the distribution of sources. Finally, at the surface S the field satisfies a homogeneous boundary condition of the form

$$\frac{\partial\psi}{\partial n} = -ik\ \delta\psi, \tag{2}$$

where δ is a certain (complex) constant of proportionality.

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As we have formulated it, this problem first arose as a simplification of the problem of radio wave propagation over that irregular and inhomogeneous interface that is our Earth. At the higher frequencies being used today, radio wave transmission is greatly affected by these irregularities, and it is becoming increasingly important to find some way of estimating the resulting field strengths.

The simplifications in the present case involve (1) a scalar wave phenomenon, and (2) a homogeneous boundary condition. But, while the remaining problem is not itself without merit, it might be remarked that both assumptions can be looked upon as reasonable approximations to the original radio wave problem. The first follows from the fact that a vertically polarized source will give rise in the main to a vertically polarized field, and a horizontally polarized source to a horizontally polarized field. Thus the vertical electric field or the vertical magnetic field or a Hertz potential, all of which satisfy the scalar wave equation, will serve to characterize the field. As for the second assumption, divers arguments that lend credence to it have been presented by Schelkunoff [1], Leontovich [2], Leontovich and Fock [3], Feinberg [4], and the present author [5]. These show that in a vertically polarized field something equivalent to Eq. (2) will be satisfied with

$$\delta \approx \frac{(\eta - 1)^{1/2}}{\eta} \approx \frac{1}{n}, \tag{3}$$

where η is the complex dielectric constant of the earth

$$\eta = \frac{\epsilon}{\epsilon_0} + i\,\frac{\sigma}{\epsilon_0\omega},$$

(ϵ/ϵ_0 being the dielectric constant and σ the conductivity) and n is the coefficient of refraction equal to $\eta^{1/2}$. Similarly in a horizontally polarized field Eq. (2) will be satisfied with

$$\delta \approx (\eta - 1)^{1/2} \approx n. \tag{4}$$

Previous authors in their attacks on the wave equation in the presence of irregular boundaries have for the most part used a perturbation method. This method was originated by Brillouin [6] and later developed by Feshbach [7] and Cabrera [8]. It begins by deriving an integral equation and then solves this by approximating the characteristic functions through known solutions to problems involving regular boundaries. However, it was pointed out by Feinberg [4] that in the special case of radio transmission this same integral equation could be reduced to a much simpler form. Subsequently, the present author in his Master's thesis [5] tried to simplify the integral equation further in the hopes that it could then be directly solved by numerical methods. The present paper is a recapitulation and a continuation of this approach.

The integral equation. According to Green's theorem, if ϕ and ψ are any two functions continuous throughout a volume V, then [9]

$$\int_V (\psi\nabla^2\phi - \phi\nabla^2\psi)\,dv = \int_{S^*} \left(\phi\,\frac{\partial\psi}{\partial n} - \psi\,\frac{\partial\phi}{\partial n}\right) da, \tag{5}$$

where S^* is the surface bounding V. Now let us consider the volume which is bounded as in Fig. 1 by the surface S and the large hemisphere-like surface Σ. Let P be some fixed point on S and Q the variable point of integration of either the volume integral or

the surface integral. Then suppose that $\psi(Q)$ satisfies Eqs. (1) and (2) while $\phi(Q)$ is the Green's function e^{ikr_2}/r_2, r_2 being the radial distance PQ. Of course, since $\phi(Q)$ as thus defined has a singularity at P, we must exclude P from the volume V by indenting it with a very small hemisphere σ.

Under these hypotheses the volume integral of Eq. (5) becomes

$$4\pi \int_V \tau(Q) \frac{e^{ikr_2}}{r_2}\, dv = 4\pi\psi_0(P), \tag{6}$$

say. This function $\psi_0(P)$ is the field that would be obtained at P if there were no earth to contend with. It is the "free-space field" due to the source distribution τ.

The surface S^* consists of three parts: Σ, σ, and S. Since ψ must satisfy the radiation condition [10], it is clear that as the radius of Σ becomes larger and larger, the integral over Σ vanishes. If the radius of σ tends to zero, then because of the singularity in $\phi(Q)$ the integral over σ approaches $2\pi\psi(P)$. Finally we know that Eq. (2) is satisfied at all points on S. Thus, after these considerations and after a little rearrangement, Eq. (5) becomes [11]

$$\psi(P) = 2\psi_0(P) + \frac{ik}{2\pi} \int_S \psi(Q) \frac{e^{ikr_2}}{r_2} \left[\delta + \left(1 + \frac{1}{kr_2}\right) \frac{\partial r_2}{\partial n} \right] da. \tag{7}$$

This as you see, is an integral equation which defines uniquely the unknown $\psi(P)$ at all points on S. We want now to simplify it with a few judicious approximations.

First, we suppose that the sources are all located on some antenna structure which is erected at a point O on S. Then we write

$$\psi_0(P) = g(P) \frac{e^{ikr_0}}{r_0}, \tag{8}$$

where r_0 is the radial distance OP. The function $g(P)$ is, in a way, the antenna pattern; it represents the gain of the antenna over an isotropic radiator at 0.

We also introduce an attentuation function $W(P)$ which is defined so that

$$\psi(P) = 2W(P) \frac{e^{ikr_0}}{r_0}. \tag{9}$$

Substituting these two equations into Eq. (7) we have

$$W(P) = g(P) + \frac{ik}{2\pi} \int_S W(Q) e^{ik(r_1+r_2-r_0)} \frac{r_0}{r_1 r_2} \left[\delta + \left(1 + \frac{1}{kr_2}\right) \frac{\partial r_2}{\partial n} \right] da. \tag{10}$$

Here now is the crux of the matter. Because of the factor $\exp ik(r_1 + r_2 - r_0)$, the integrand in this last equation oscillates very rapidly from one point on S to another. This fact implies that we might use Kelvin's principle of stationary phases; that is, we could approximate the integral by summing up only those contributions which come from the neighborhoods of points where the phase of the integrand is stationary. These points in general form a discrete set which are exactly the "points of reflection" used in geometrical optics, and indeed, doing this should give us precisely the geometrical optics solution. It may be remarked that this very solution has already been used with some success by McPetrie and Ford [12] and by Shelleng, Burrows, and Ferrell [13] to analyze some actual data, while Keller and Keller [14] have devised several formulas to be used in this connection.

However, we cannot in our problem use this particular approximation, for when the geometrical rays are at nearly grazing angles of incidence or, even worse, when we must consider diffraction regions, then it is not nearly accurate enough. The reason for this is that at points between O and P the phase, while not necessarily stationary, is still but slowly varying. Thus to improve upon the geometrical optics solution we must reduce the integral in Eq. (10) not to the sum of discrete contributions, but to a line integral from O to P.

To do this we first project everything onto a horizontal plane S'. Then the integral can be written in the form

$$I = \int_{S'} F(Q')e^{ik(r_1'+r_2'-r_0')} \frac{da'}{r_1'r_2'},$$

where the primes are used to represent horizontal projections. Now let us construct on S' a set of elliptic coordinates (u, v) defined by

$$r_0' \cosh u = r_2' + r_1' , \qquad r_0' \cos v = r_2' - r_1' .$$

The differential area is $r_2'r_1' \, du \, dv$ and the integral becomes

$$I = \int_{-\infty}^{\infty} \exp\,[ikr_0'(\cosh u - 1)]\, du \int_0^{\pi} F(u, v)\, dv.$$

This, as an integral in u, has the phase kr_0' (cosh $u - 1$) which is stationary only at the point $u = 0$. Immediately, therefore, we can write down the approximation [15]

$$I \approx \left(\frac{2\pi}{kr_0'}\right)^{1/2} e^{i\pi/4} \int_0^{\pi} F(0, v)\, dv.$$

This approximation is good, of course, only if $\int_0^{\pi} F(u, v)\, dv$ is a slowly varying function of u. But this will always be the case if S is smooth enough. If, on the other hand, there are projections on S which are away from the line OP but which nevertheless contain points of reflection, then surely these must also be taken into account in evaluating the integral. In what follows we shall assume that no such projections exist.

The line $u = 0$, $0 \leq v \leq \pi$, to which we have reduced the surface integral, is the line segment $O'P'$. Adopting at this point a more suitable one dimensional notation, let us denote the distance r_0' by the letter x and the distance of the point $(0, v)$ from O by s. Then,

$$s = \tfrac{1}{2}x(1 - \cos v),$$

so that our approximation becomes

$$\left(\frac{2\pi}{kx}\right)^{1/2} e^{i\pi/4} \int_0^{x} F(s) \frac{ds}{[s(x - s)]^{1/2}}.$$

The final form of our integral equation can be further simplified by making several more approximations in $F(s)$. Thus we assume that da/da' is close to unity, that $1/kr_2$ $\partial r_2/\partial n$ is negligible, and that whenever r_0, r_1, r_2 appear in the modulus of $F(s)$ they can be replaced by the horizontal distances x, s, $x - s$, respectively. Equation (10) then becomes

$$W(x) = g(x) - e^{-i\pi/4}\left(\frac{k}{2\pi}\right)^{1/2} \int_0^{x} W(s)\left(\delta + \frac{\partial r_2}{\partial n}\right)e^{ik(r_1+r_2-r_0)}\left[\frac{x}{s(x - s)}\right]^{1/2} ds. \qquad (11)$$

This is the integral equation we are proposing. It is, so to speak, an approximate representation of our problem which has the advantage of simplicity while still retaining most of the characteristics upon which one intuitively feels the problem should depend. Actually, the reduction to one dimension is not an unfamiliar concept. Experimental and theoretical investigators have for a long time drawn profiles of the earth and attempted to correlate the profile with measured field strengths. Following our own

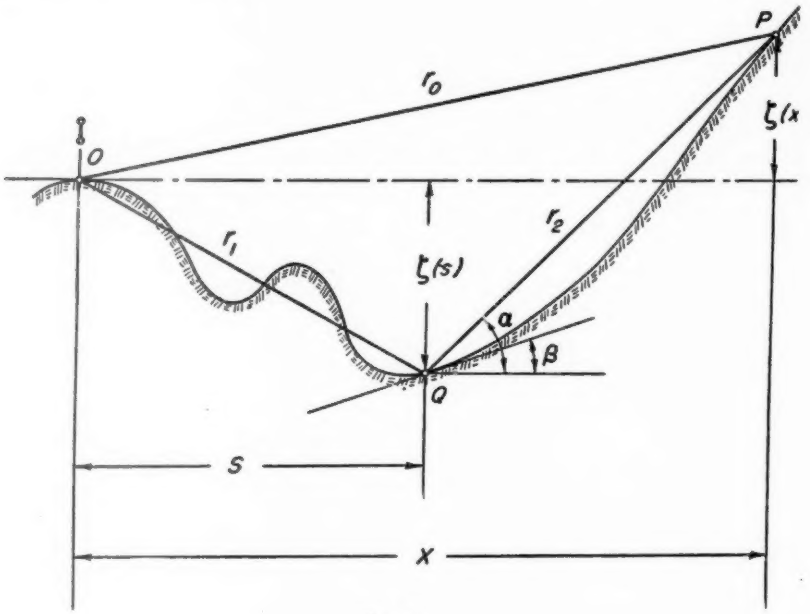

FIG. 2.

proposal, we would draw a profile similar to the one in Fig. 2; compute from it the quantities r_0, r_1, r_2, $\partial r_2/\partial n$; substitute these in Eq. (11); and then, by using suitable numerical methods, solve for $W(x)$.

Indeed, let us represent the profile in question by the function $\zeta(x)$. This function will be the elevation of the earth above a datum plane passing through the point O. Then we can write

$$r_0 = [x^2 + \zeta^2(x)]^{1/2} \approx x + \frac{\zeta^2(x)}{2x}$$

and similar expressions for r_1 and r_2. A little algebraic manipulation of these expressions gives us

$$r_1 + r_2 - r_0 \approx \frac{sx}{2(x - s)} \left[\frac{\zeta(s)}{s} - \frac{\zeta(x)}{x} \right]^2. \tag{12}$$

Similarly,

$$\frac{\partial r_2}{\partial n} = \sin(\beta - \alpha) \approx \zeta'(s) - \frac{\zeta(x) - \zeta(s)}{x - s}. \tag{13}$$

The field above the earth. We have thus found a procedure for evaluating $W(x)$ and hence the field ψ at all points of the earth's surface. It remains for us to show how this can be used to find the field at points in the space above this surface.

Using Green's theorem again together with Eq. (1) we find

$$4\pi \int_V \tau(Q) \frac{e^{ikR_2}}{R_2} dv = 4\pi\psi(P) - \int_S \left[\psi(Q) \frac{\partial}{\partial n} \frac{e^{ikR_2}}{R_2} - \frac{e^{ikR_2}}{R_2} \frac{\partial \psi(Q)}{\partial n} \right] da,$$

where this time P is in the space above S. The quantity R_2 is the radial distance PQ; we use the capital letter to stress the fact that P has been raised off the earth. Note that to exclude P from the integral of Green's theorem we now must construct a complete sphere about P rather than a hemisphere. It is for this reason that the $2\pi\psi(P)$ has changed to $4\pi\psi(P)$.

Now this equation, too, contains a surface integral whose integrand oscillates rapidly everywhere except possibly between O and P. Hence we can follow exactly the same steps we used before to change the integral to an approximate one-dimensional integral. If we replace the volume integral by $\psi_0(P)$ and use Eq. (2) to dispose of $\partial\psi/\partial n$, we shall finally arrive at the equation

$$\psi(P) = \psi_0(P) - e^{-i\pi/4}\left(\frac{k}{2\pi}\right)^{1/2} \int_0^x W(s)\left(\delta + \frac{\partial R_2}{\partial n}\right) e^{ik(r_1+R_2)} \left[\frac{x}{s(x-s)}\right]^{1/2} ds. \tag{14}$$

The quantities $r_1 + R_2$ and $\partial R_2/\partial n$ can be approximated by suitable modifications of Eqs. (12) and (13).

A plane earth. Although Eqs. (11) and (14) were derived largely to provide numerical solutions, it is perhaps of some interest to examine a few special cases which can be solved analytically.

One such case is that of a plane, homogeneous earth with an isotropic antenna at a height h_1 above the point O. Then $\partial r_2/\partial n = 0$, $r_1 + r_2 - r_0 = 0$, and

$$\psi_0(x) = (x^2 + h_1^2)^{-1/2} \exp\left[ik(x^2 + h_1^2)^{1/2}\right] \approx \frac{1}{x} \exp\left[ik\left(x + \frac{h_1^2}{2x}\right)\right]$$

or

$$g(x) \approx \exp(ikh_1^2/2x). \tag{15}$$

Now it may be argued that this last approximation can hardly be valid since at the point O where $x = 0$, the error becomes infinite. But while this will certainly affect the solution near O it will actually have very little affect further away. This is because the phase of the error also becomes infinite causing the integrated error to be small.

Introducing, then, these quantities into Eq. (11) we have

$$W(x) = \exp(ikh_1^2/2x) - e^{-i\pi/4}\delta\left(\frac{k}{2\pi}\right)^{1/2} \int_0^x W(s)\left[\frac{x}{s(x-s)}\right]^{1/2} ds. \tag{16}$$

This we can simplify further with the use of Sommerfeld's complex numerical distance. We define

$$\rho = i\frac{k\delta^2}{2} x \qquad \nu = i\frac{k\delta^2}{2} s,$$

whereupon Eq. (16) becomes

$$W(\rho) = e^{-a^2/4\rho} + i\pi^{-1/2} \int_0^\rho W(\nu)\left[\frac{\rho}{\nu(\rho - \nu)}\right]^{1/2} d\nu, \tag{17}$$

where $a = kh_1\delta$.

This integral equation can now be solved in any of several ways—by using the Liouville-Neumann series or by a method similar to that used in solving Abel's integral equation. We shall find it most convenient to use Laplace transforms. This we do first by defining

$$U(\rho) = \rho^{-1/2} W(\rho); \tag{18}$$

then Eq. (17) becomes

$$U(\rho) = \rho^{-1/2} e^{-a^2/4} + i\pi^{-1/2} \int_0^\rho \frac{U(\nu)}{(\rho - \nu)^{1/2}} d\nu. \tag{19}$$

In this equation the integral is of the *Faltung* or convolution type, so that when we apply the Laplace transformation to both sides we obtain [16]

$$L\{U\} = \left(\frac{\pi}{p}\right)^{1/2} e^{-ap^{1/2}} + ip^{-1/2} L\{U\}$$

or, solving for $L\{U\}$,

$$L\{U\} = \frac{\pi^{1/2} e^{-ap^{1/2}}}{p^{1/2} - i} = \left(\frac{\pi}{p}\right)^{1/2} e^{-ap^{1/2}} + i\left(\frac{\pi}{p}\right)^{1/2} \frac{e^{-ap^{1/2}}}{p^{1/2} - i}. \tag{20}$$

It only remains for us to find the inverse transform of this function. But the first term is the transform of $\rho^{-1/2} e^{-a^2/4}$, while the second term we can write as $i(\pi/p)^{1/2} f(p^{1/2})$ which is the Laplace transform of [17]

$$i\rho^{-1/2} \int_0^\infty e^{-t^2/4\rho} V(t)\, dt,$$

where

$$V(t) = L^{-1}\{f(p)\} = L^{-1}\left\{\frac{e^{-ap}}{p - i}\right\} = \begin{cases} 0 & t < a, \\ e^{i(t-a)} & t \geq a. \end{cases}$$

(This last, of course, assumes that a is real. But the functions that we obtain in the end will all be analytic in a and hence valid also for complex values.) Thus

$$\begin{aligned} U(\rho) &= \rho^{-1/2} e^{-a^2/4} + i\rho^{-1/2} \int_a^\infty \exp\left[-\frac{t^2}{4a} + i(t - a)\right] dt \\ &= \rho^{-1/2} e^{-a^2/4} + i\pi^{1/2} e^{-\rho - ia} \operatorname{erfc}\left(-i\rho^{1/2} + \frac{a}{2\rho^{1/2}}\right), \end{aligned} \tag{21}$$

where

$$\operatorname{erfc}(x) = \frac{2}{\pi^{1/2}} \int_x^\infty e^{-u^2} du.$$

Or, if we define

$$w = \rho\left(1 + \frac{ia}{\rho}\right)^2 = \rho\left(1 + \frac{h_1}{\delta x}\right)^2, \tag{22}$$

we have finally

$$W(\rho) = \rho^{1/2}U(\rho) = \exp\,[ikh_1^2/2x][1 + i(\pi\rho)^{1/2}e^{-w}\,\mathrm{erfc}\,(-iw^{1/2})]. \tag{23}$$

To complete our analysis, suppose that we raise the receiving antenna to a height h_2. From Eq. (14) together with the usual approximations for R_2 and $\partial R_2/\partial n$, we have

$$\begin{aligned}\psi(P) &= \frac{e^{ikR_0}}{R_0}\\ &+ i\left(\frac{\rho}{\pi}\right)^{1/2} e^{ikx} \int_0^\rho \frac{W(\nu)}{\nu^{1/2}}\left[1 - \frac{ikh_2}{2(\rho - \nu)}\right](\rho - \nu)^{-1/2}\exp\,[-k^2h_2^2\delta^2/4(\rho - \nu)]\,d\nu,\end{aligned} \tag{24}$$

where R_0 is the radial distance from the isotropic antenna to the point P.

In order to evaluate this integral we shall again make use of the *Faltung* theorem. Denoting the integral by I and remembering Eq. (20),

$$\begin{aligned}L\{I\} &= L\{\rho^{-1/2}W(\rho)\}L\left\{\left(1 - \frac{ikh_2}{2\rho}\right)\rho^{-1/2}\exp\,(-k^2h_2^2\delta^2/4\rho)\right\}\\ &= i\pi\exp\,[-k(h_1 + h_2)\,\delta p^{1/2}]\left[\frac{1}{p^{1/2}} - \frac{2}{p^{1/2} - i}\right].\end{aligned} \tag{25}$$

Thus the Laplace transform of I is represented as the sum of two terms, the first of which is the transform of $i(\pi/\rho)^{1/2}\exp\,[-k^2(h_1 + h_2)^2\delta^2/4\rho]$, while the second is exactly the same transform as that in Eq. (20) except for the factor $-i2\pi^{1/2}$ and except that h_1 has been replaced by $h_1 + h_2$. Therefore we have immediately

$$\begin{aligned}\psi(P) &= \frac{e^{ikR_0}}{R_0}\\ &- \frac{1}{x}\exp\,\{ik[x + (h_1 + h_2)^2/2x]\}\{1 - 2[1 + i(\pi\rho)^{1/2}e^{-w}\,\mathrm{erfc}\,(-iw)^{1/2}]\},\end{aligned} \tag{26}$$

where now

$$w = \rho\left(1 + \frac{h_1 + h_2}{\delta x}\right)^2. \tag{27}$$

The problem of electromagnetic radiation from a vertical Hertzian dipole over a plane and homogeneous earth was first successfully attacked by Sommerfeld in 1909, and since then many authors have treated its various aspects. To compare our Eq. (26) with the classical solutions, we quote here an approximation due to Norton: [18]

$$\psi(P) = \frac{e^{ikR_0}}{R_0} - \frac{e^{ikR_0'}}{R_0'}\{1 - 2[1 + i(\pi\rho)^{1/2}e^{-w}\,\mathrm{erfc}\,(-iw)^{1/2}]\},$$

where $\psi(P)$ is the Hertz potential at P,

$$\rho = ikR_0'/2n^2,$$

$$w = \rho[1 + n(h_1 + h_2)/R_0']^2,$$

n is the coefficient of refraction that we mentioned in Eq. (3), and R_0' is the radial distance from the dipole to the image of the point P,

$$R_0'^2 = x^2 + (h_1 + h_2)^2.$$

The agreement, it will be noticed, is almost exact.

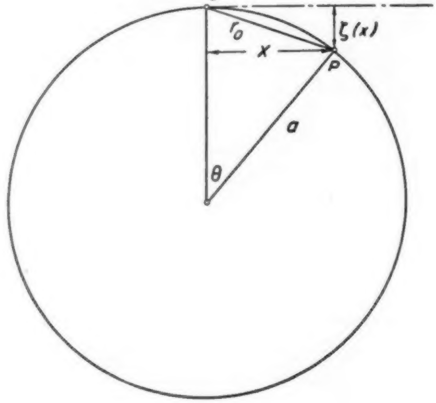

FIG. 3.

A spherical earth. As a last example let us consider an isotropic antenna at the surface of a spherical, homogeneous earth. If the sphere is of radius a, then from Fig. 3 we have

$$\zeta(x) = (a^2 - x^2)^{1/2} - a \approx -\frac{x^2}{2a}. \tag{28}$$

Substituting this approximation into Eqs. (12) and (13) gives us

$$r_1 + r_2 - r_0 = \frac{sx(x - s)}{8a^2},$$

$$\frac{\partial r_2}{\partial n} = \frac{x - s}{2a}. \tag{29}$$

Now we introduce the so-called natural units ξ, ζ, and γ, which are defined by

$$\xi = xa^{-2/3}(k/2\pi)^{1/3},$$

$$\zeta = sa^{-2/3}(k/2\pi)^{1/3}, \tag{30}$$

$$\gamma = i/(ka)^{1/3},$$

and further make the substitution

$$W(\zeta) = \xi^{-1/2} e^{i(\pi/12)\xi^3} U(\xi). \tag{31}$$

Then our problem of a spherical earth becomes immediately one of solving the integral equation

$$\begin{aligned} U(\xi) &= \xi^{-1/2} \exp\left[-i\left(\frac{\pi}{12}\right)\xi^3\right] - \\ &- \int_0^\xi U(\zeta) \exp\left[-i\left(\frac{\pi}{12}\right)(\xi-\zeta)^3\right]\left[\frac{e^{i\pi/4}}{\gamma(2\pi)^{1/3}} + \frac{1}{2} e^{-i\pi/4}(\xi-\zeta)\right](\xi-\zeta)^{-1/2}\, d\zeta. \end{aligned} \tag{32}$$

Again we have a *Faltung* integral. If we denote by $u(p)$ the Laplace transform of $U(\xi)$ and by $v(p)$ the transform of the function $\xi^{-1/2}e^{-i(\pi/12)\xi^3}$, then taking the Laplace transformation of both sides of Eq. (32) and making use of the *Faltung* theorem and the rule for differentiating a transform, we find easily

$$u(p) = \frac{v(p)}{1 + e^{i\pi/4}(2\pi)^{-1/3}\gamma^{-1}v(p) - (1/2)e^{-i\pi/4}v'(p)}. \tag{33}$$

And now to find $U(\xi)$ we need only to use the complex inversion formula

$$U(\xi) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} u(p)e^{p\xi}\, dp. \tag{34}$$

Leaving aside, for the moment, all questions of existence and convergence we remark that this integral can be evaluated from a knowledge of the poles of $u(p)$. As we shall show later, $v(p)$ is an integral function so that the poles of $u(p)$ are all at the zeros of the denominator in Eq. (33). It follows that if these zeros are at the points $p = p_n$, $n = 0, 1, \cdots$, we shall finally have

$$U(\xi) = \sum_{n=0}^{\infty} \frac{v(p_n)e^{p_n\xi}}{e^{i\pi/4}(2\pi)^{-1/3}\gamma^{-1}v'(p_n) - (1/2)e^{-i\pi/4}v''(p_n)}. \tag{35}$$

The properties of $v(p)$. We have defined $v(p)$ by the integral

$$v(p) = \int_0^\infty \xi^{-1/2}e^{i(\pi/12)\xi^3}e^{-p\xi}\, d\xi \tag{36}$$

which, however, converges only for $\mathrm{Re}\, p \geq 0$. If we make the transformation

$$\xi = (12/\pi)^{1/3}e^{-i\pi/6}t, \tag{37}$$

then it follows that

$$v(p) = (12/\pi)^{1/6}e^{-i\pi/12}k[(12/\pi)^{1/3}e^{-i\pi/6}p], \tag{38}$$

where

$$k(z) = \int_0^\infty t^{-1/2}e^{-t^3-zt}\, dt. \tag{39}$$

Since this last integral is analytic for all z it defines the analytical continuation of $v(p)$ to the entire p-plane.

Obviously $k(z)$ is regular everywhere and is therefore an integral function. It has the Taylor series expansion

$$k(z) = \int_0^\infty t^{-1/2}e^{-t^3} \sum_{n=0}^\infty \frac{(-zt)^n}{n!}\,dt = \frac{1}{3}\sum_{n=0}^\infty \frac{\Gamma(n/3 + 1/6)}{\Gamma(n + 1)}(-z)^n. \tag{40}$$

Now from this Taylor series, it is possible to derive an asymptotic expansion valid for large z. We need to know that the function $\Gamma[(z/3) + (1/6)]/\Gamma(z + 1)$ has poles at $z = -\frac{1}{2}, -3\frac{1}{2}, \cdots$, and that everywhere to the right of the imaginary axis it has the asymptotic expansion

$$\frac{\Gamma(z/3 + 1/6)}{\Gamma(z + 1)} \sim \pi^{1/2}\frac{2^{4/3}}{3^{1/2}}\left(\frac{2^{2/3}}{3}\right)^z \frac{1}{\Gamma(2z/3 + 4/3)}\left[1 + 0\left(\frac{1}{z}\right)\right].$$

Then it can be shown that [19]

$$k(z) \sim \begin{cases} \pi^{1/2}z^{-1/2} & -\dfrac{2\pi}{3} < \arg z < \dfrac{2\pi}{3}, \\[2ex] \pi^{1/2}[iz^{-1/2}\exp\{i(2/3^{3/2})z^{3/2}\} + z^{-1/2}] & \dfrac{2\pi}{3} \le \arg z \le \pi, \\[2ex] \pi^{1/2}[iz^{-1/2}\exp\{i(2/3^{3/2})z^{3/2}\} - z^{-1/2}] & \pi \le \arg z \le \dfrac{4\pi}{3}. \end{cases} \tag{41}$$

From this it follows that the zeros of $k(z)$ lie on or near the radials $e^{i2\pi/3}$ and $e^{i4\pi/3}$ since it is there that the exponentials of Eq. (41) have an absolute value equal to one. If we return to a consideration of the original function $v(p)$ we see from Eq. (38) that its zeros will lie along the radials $e^{i3\pi/2}$ and $e^{i5\pi/6}$. In fact if we let

$$p = \sigma e^{i3\pi/2} = \tau e^{i5\pi/6} \tag{42}$$

then for small arguments of σ and τ we have respectively

$$\begin{aligned} v(p) &\sim \pi^{1/2}e^{-i\pi/4}\sigma^{-1/2}\{\exp[i(4/3\pi^{1/2})\sigma^{3/2}] - \exp(-i\pi/2)\}, \\ v(p) &\sim \pi^{1/2}e^{i\pi/12}\tau^{-1/2}\{\exp[-i(4/3\pi^{1/2})\tau^{3/2}] - \exp(i\pi/2)\}, \end{aligned} \tag{43}$$

and further

$$\begin{aligned} v'(p) &\sim -2e^{-i\pi/4}\exp[i(4/3\pi^{1/2})\sigma^{3/2}], \\ v'(p) &\sim -2e^{-i\pi/4}\exp[-i(4/3\pi^{1/2})\tau^{3/2}], \\ v''(p) &\sim -4\pi^{-1/2}e^{i5\pi/12}\tau^{1/2}\exp[-i(4/3\pi^{1/2})\tau^{3/2}]. \end{aligned} \tag{44}$$

We shall need one more property of the function $v(p)$. It can easily be shown, after integrating Eq. (39) by parts, that

$$6k''' + 2zk' + k = 0. \tag{45}$$

On the other hand

$$\frac{d}{dz}\left[kk'' - \frac{1}{2}(k')^2 + \frac{z}{6}k^2\right] = \frac{1}{6}k[6k''' + 2zk' + k] = 0,$$

so that

$$kk'' - \frac{1}{2}(k')^2 + \frac{z}{6}k^2 = \text{const.} \tag{46}$$

The constant of integration, which can be determined by evaluating Eq. (40) and its derivatives at $z = 0$, is equal to $\pi/6$.

From this we can conclude that whenever $k(z) = 0$, then $k'(z) = \pm i(\pi/3)^{1/2}$. Or again, whenever $v(p) = 0$, then

$$v'(p) = \pm 2e^{i\pi/4}. \tag{47}$$

Furthermore, it follows from Eqs. (43) and (44) that at those zeros lying along the radial $e^{i3\pi/2}$ it is the positive sign that is valid, while along the radial $e^{i5\pi/6}$ it is the negative sign.

The field. Having thus determined the salient properties of the function $v(p)$ we are now in a position to discuss the properties of $u(p)$ and its inverse transformation.

First of all, it can be shown from Eq. (33) and from the asymptotic expansions of $v(p)$ that, except in the neighborhood of its poles, $u(p) = 0(p^{-1/2})$. From this it follows [20] that the integral of Eq. (34) converges, that it represents the function $U(\xi)$, and that it can be expanded in the series of Eq. (35). Thus this series is established rigorously as the solution to Eq. (32).

Now ordinarily γ is a very small quantity so that we shall expect the zeros of the denominator in Eq. (33) to be very near the zeros of $v(p)$. Thus there will be two sets of zeros: those in the vicinity of the radial $e^{i3\pi/2}$ and those near the radial $e^{i5\pi/6}$. And as a matter of fact, it follows from Eq. (47) that those zeros of $v(p)$ lying near the radial $e^{i3\pi/2}$ are exactly zeros of the denominator. But because they are zeros of $v(p)$ they can contribute nothing to the series of Eq. (35) since the numerator of each term contains a factor $v(p_n)$.

Thus we are left with only those zeros that lie near the radial $e^{i5\pi/6}$. If we write $p_n = \tau_n e^{i5\pi/6}$ and make use of the asymptotic approximations in Eqs. (43) and (44), then finding the zeros becomes equivalent to solving the equation

$$e^{-i(4/3\pi^{1/2})\tau_n^{3/2}} = -i\,\frac{1 + e^{-i\pi/6}(\pi/4)^{1/6}/\gamma\tau_n^{1/2}}{1 - e^{-i\pi/6}(\pi/4)^{1/6}/\gamma\tau_n^{1/2}} \tag{48}$$

or alternatively

$$\tan\left(\frac{\pi}{4} - \frac{2\tau_n^{3/2}}{3\pi^{1/2}}\right) = -e^{i\pi/3}\left(\frac{\pi}{4}\right)^{1/6}\frac{1}{\gamma\tau_n^{1/2}}. \tag{49}$$

And now we can write down the final answer. But before we do let us note that because of the geometry of the sphere

$$\begin{aligned} kr_0 + \frac{\pi}{12}\xi^3 &= k\left(r_0 + \frac{x^3}{24a^2}\right) \\ &= ka\left(\theta - \frac{\theta^3}{24} + \cdots + \frac{\theta^3}{24} + \cdots\right) \\ &\approx ka\theta. \end{aligned} \tag{50}$$

Then it immediately follows from Eqs. (9), (31), (35), (48), and (50) that

$$\psi(P) = \frac{e^{ika\theta}}{r_0} 2e^{i\pi/4}\xi^{1/4} \sum_{n=0}^{\infty} \frac{\exp(-\tau_n e^{-i\pi/6}\xi)}{e^{i\pi/3}\pi^{-1}\tau_n - (2\pi)^{-2/3}\gamma^{-2}}. \tag{51}$$

Once again we have a solution which we can compare with the classical solutions for electromagnetic radiation; and happily enough we shall find much that is similar. Indeed, van der Pol and Bremmer [21] in finding the Hertz potential excited by a vertical electric dipole at the surface of a spherical, homogeneous earth, have suggested an equation which is—even to the extent of defining δ as in Eq. (3)—identical with Eq. (51). Furthermore, Eq. (49) is identical with their so-called tangent approximation to define the τ_n. It is of some interest to note that Fock [22] in still another attack on this same problem agrees even more directly with our results. He derives a contour integral which is very similar in form to Eq. (34) and then deduces a series expansion which is again identical with Eq. (51).

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—NOTES—

A METHOD OF VARIATION FOR FLOW PROBLEMS—II*

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Summary. The method of variation of reference [1] is developed afresh in a slightly different manner which enables the main principle used in [1] to be derived directly and also makes the actual calculations much simpler. It is shown how a variety of problems concerning aerofoils possessing minimal properties may be reduced to the solution of integro-differential equations which determine the mapping of the aerofoil onto a circular region. It is briefly indicated how the method may be extended to three dimensional flows.

1. Introduction. In [1] the author has given an elementary method of variation suitable for treating a type of extremal problems suggested by two-dimensional aerofoil theory. Briefly, the method is to make small elliptic bulges in the boundary and after calculating the changes of the flow functions to equate to zero the variation of the functional which is to be minimized, for the case of infinitely flat ellipses. This process was justified by appealing to the fact that for such flat bulges the velocity changes as well as the geometrical changes were in effect infinitesimal. Such a physical argument being not quite convincing the author also tested the principle by showing that *all* small bulges in the *hodograph* plane gave equivalent results. It will however be seen that it is quite easy to prove that for infinitesimal velocity changes, although not for general perturbations of the boundary, the first variations may be equated to zero.

The results of the method appear in a conveniently compact form viz. as integro-differential equations (sometimes just differential equations) from which the mapping of the aerofoil on the standard unit circle may be determined. In some problems it is necessary to make auxiliary restrictions on the aerofoil, such as limitations on the chord or the velocity in the field.

In three dimensional fields the device of conformal mapping is of course not available but, analogous to the above equations for aerofoil problems, the method yields certain relations between geometrical and field properties on the boundary of the field. A discussion of the determination of the field from such conditions is left for a future paper.

2. The method of variation. In [1] the following principle was treated as physically obvious:—

If a functional of the geometry of a closed curve and the velocities of an associated hydrodynamic field is maximized by a certain curve, this functional is stationary for all variations in which both physical coordinates and the velocities are changed infinitesimally.

On the other hand small bulges giving rise to finite changes of velocity will change the functional by a small quantity of the *first* order. This situation may be illustrated by the following simple case. It is easily shown that the ratio of the area of an ellipse to the strength of the doublet, giving an equivalent disturbance in the stream at large distances, depends on the shape of the ellipse. Let it be admitted that there is some aerofoil problem in which the ratio of area to disturbance at infinity is stationary (c.f. §5).

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Then by taking two small ellipses of the same area it would be possible to find an equation corresponding to zero variation of area but a non-zero disturbance at infinity. Equally one could determine a variation corresponding to no disturbance but non-zero change of area. The ratio cannot be stationary for such variations. Further illustration may be found in the detailed calculations of [1] for small elliptic bulges. It is also interesting to note that, since velocity changes are *finite*, small elliptic bulges give something more than a second variation and so the sign of the changes is rather strong evidence, although not proof, of a true maximum or minimum.

The principle at the beginning of this paragraph will now be embodied in a simple lemma capable of direct proof.

Main lemma. If ψ satisfies a second order partial differential equation of elliptic type together with the boundary conditions on a simple closed curve C (and suitable auxiliary conditions) whilst I is a functional of C, in the geometrical sense, and of the velocities at points in the field and on C, then, for variations giving infinitesimal velocity changes everywhere, $\delta I = 0$ if I is a maximum or minimum.

Proof. If a small perturbing stream function is added to ψ it is readily seen that $\delta\psi$ satifies a homogeneous linear partial differential equation and so, if $\delta\psi$ is a solution so is $-\delta\psi$. Now the new boundary C': $\psi + \delta\psi = 0$ is to be derived from C by drawing a normal to C of length

$$\delta n = -\left[\delta\psi \Big/ \frac{\partial\psi}{\partial n}\right]_c = -\left(\frac{\delta\psi}{q_s}\right)_c. \tag{2.1}$$

Except at the isolated stagnation points the velocity q_s is not zero so the perturbation is infinitesimal provided $\delta\psi$ is made zero at such points.

Geometrical changes may be expressed linearly in terms of δn. For example, if A is the area enclosed by C whilst L is the length of C and ds the line element

$$\delta A = \int_c \delta n \, ds, \tag{2.2}$$

$$\delta L = \int_c K \, \delta n \, ds, \tag{2.3}$$

where K is the curvature of C.

It is therefore clear that both the velocities at a *given* point and geometrical changes are linear in $\delta\psi$. This is also true of velocities on the variable boundary C and for the present purpose this is more important.

Thus, the variation of velocity on $\psi = 0$ may be found as the sum of :

(i) $\partial/\partial n(\delta\psi)$ calculated at C,

(ii) the change in the undisturbed field due to displacing C a distance δn (i) is clearly linear in $\delta\psi$ and (ii) is linear in δn and so again in $\delta\psi$. If $(\partial^2\psi/\partial n^2)_C$ were zero then the contribution would be zero to the first order. In fact, for elliptic equations, this case is easily shown never to arise.

If therefore I is a maximum or minimum the existence of a $-\ \delta I$ for every $+\delta I$ shows immediately that for such variations $\delta I = 0$ as in the better known geometrical problems of the calculus of variations. Simple as the proof appears the result is not trivial since it implies such results as those of §5 [1] which are difficult to derive by direct calculation.

3. Laplace's equation in two variables. Let z be the aerofoil plane, ζ that of the unit circle and t that of the strip. For non-circulating flow, U being the stream velocity, the complex potential is

$$w = Ut = U(\zeta + \zeta^{-1}). \tag{3.1}$$

A suitable perturbing function is

$$\delta w = \mu^2 U \sum_1^\infty a_n \zeta^{-n}, \tag{3.2}$$

where μ is small and the a_n are real so that $I(\delta w) = 0$ on the real axis.

If the image in the ζ-plane of the disturbed boundary streamline C' is given by the vector $\zeta = e^{i\theta}[1 + \delta R]$ it follows that

$$\delta R = \frac{\mu^2 \eta(\theta)}{2 \sin \theta}, \tag{3.3}$$

where

$$\eta(\theta) = \sum a_n \sin n\theta.$$

Following the method of the previous section, the change of velocity is the sum $\delta_1 v + \delta_2 v$, where

$$\delta_1 v = \mu^2 \sum n a_n \sin n\theta / 2 \sin \theta, \tag{3.4}$$

$$\delta_2 v/v = -\delta R(1 + \alpha) \tag{3.5}$$

with

$$\alpha = R\left(\zeta \frac{d^2 z}{d\zeta^2} \Big/ \frac{dz}{d\zeta}\right)_c,$$

(3.5) being most conveniently found by writing

$$\left| \frac{dw}{dz} \right|_{c'} = \left| \frac{dw}{d\zeta} \right|_{c'} \Big/ \left| \frac{dz}{d\zeta} \right|_{c'}$$

and expanding numerator and denominator.

If C' is mapped on the unit circle $|\zeta'| = 1$ in a new plane the complex potential is known in two forms and by comparison of both it and the velocities on C' found from the two methods it is quite easy to establish the relation.

$$\log \left| \frac{d\zeta}{d\zeta'} \right| = \Delta\theta \cot \theta' + \frac{\delta_2 v(\theta')}{v(\theta')},$$

where

$$2 \sin \theta' \Delta\theta = \mu^2 \sum a_n \cos n\theta' \qquad \text{and} \qquad \zeta'_c = e^{i\theta'}.$$

These relations together with Poissons' formula for the function $\log (d\zeta/d\zeta')$ regular in $|\zeta'| \geq 1$ and with the real part given on $|\zeta'| = 1$ determine the mapping of the new region on the old. For applications however 3.3, 3.4, 3.5 are sufficient and involve only one linear transformation of $\eta(\theta)$ whereas the explicit mapping requires three such transformations.

4. Aerofoils minimizing a surface integral. Let A be the area contained in C a symmetrical profile with axis along the stream and suppose

$$I = \int_C F(v)\, ds,$$

where v is the velocity on the boundary. Then $J \equiv IA^{-1/2}$ is non-dimensional and it has been shown in [1] that for $F(v) = kv$ with k constant J is a minimum for the circle. In fact I is not changed by conformal mapping between the aerofoil and plane of the circle but depends only on the potential function. The problem therefore reduces to making A as large as possible under conditions for which the circle is well known to be the result. It is natural to assume the existence of solutions for more general F and the method of variation shows that these must satisfy a certain integro-differential relation. From (3.5) since the equipotentials are normal to C it readily follows that $\delta(ds) = -\delta_2 v/v$ and so

$$ds_1 = d\theta \left| \frac{dz}{d\zeta} \right|_c \{1 + \delta R(1 + \alpha)\} \tag{4.1}$$

is the length of the element of the perturbed arc. Then $\delta I = \int F'(v)\delta v\, ds + \int F(v)\, \delta(ds)$ becomes, after using (3.4), (3.5) and (4.1),

$$\delta I = \int F'(v)\{\delta_1 v/v - \delta R(1 + \alpha)\} 2 \sin\theta\, d\theta + \int F(v)\, \delta R(1 + \alpha) \frac{2 \sin\theta\, d\theta}{v}. \tag{4.2}$$

Again

$$\delta A = \int \delta R \left| \frac{dz}{d\zeta} \right|^2 d\theta = 2\mu \int \eta(\theta) \frac{\sin\theta\, d\theta}{v^2}, \tag{4.3}$$

so that if $\delta J = 0$, which implies

$$\delta I = K \delta A \qquad \text{with} \qquad K = \frac{1}{2} \frac{I}{A},$$

the condition for a minimal solution may be written

$$\int_0^\pi H'(\theta)\eta(\theta)\, d\theta = \int_0^\pi F'(v)\delta_1 v_\theta\, d\theta, \tag{4.4}$$

where

$$H = \int^\theta \{[1 + \alpha][F'(v) - F(v)/v] + 2k \sin\theta/v^2\}\, d\theta. \tag{4.5}$$

Then $\delta_1 v_\theta$, the change in the plane of the circle is with $\nu(\theta) = \eta'(\theta)$ given by

$$\delta_1 v_\theta = \frac{\mu^2}{\pi} \int_0^\pi \frac{\sin\theta}{\cos t - \cos\theta} \nu(t)\, dt \tag{4.6}$$

in which to make the operation on ν possible it is supposed for example that $\nu(t)$ possesses a derivative.
Since η vanishes at the stagnation points at the ends of the aerofoil 4.4 becomes after an integration by parts

$$\int_0^\pi F'(v)\left[\frac{1}{\pi}\int_0^\pi \frac{\sin\theta}{\cos t - \cos\theta}\nu(t)\,dt\right]d\theta + \int_0^\pi H(t)\nu(t)\,dt = 0. \tag{4.7}$$

If now $\nu(t)$ vanishes except in a small interval (4.7) gives

$$\frac{1}{\pi}\int_0^\pi \frac{F'\{v(\theta)\}\sin\theta}{\cos t - \cos\theta}\,d\theta + H(t) = 0, \tag{4.8}$$

where H is given by (4.5).

In the case $F = kv$ H simplifies greatly and (4.8) gives

$$\frac{2k\sin t}{v^2(t)} = \frac{1}{\pi}\frac{d}{dt}\left\{\log\left(\frac{1-\cos t}{1+\cos t}\right)\right\} = \frac{2}{\pi\sin t}.$$

Hence $v \propto \sin t$ and $|\,dz/d\zeta\,|$ = constant showing that the solution must be a circle. The following approximations are suggested in other cases.

(*a*) If the body is nearly circular α is small and the relation Eq. (4.8) is satisfied approximately by

$$F(v) = v + a(e^{bv^2} - 1) \qquad ve^{bv^2} = 2\,a\sin\theta,$$

where b is small. This gives *one* family of solutions.

(*b*) For thin flat aerofoils $(1 + \alpha)$ is small except at the ends and if $F(0) = 0$ the term $F - vF' = 0(\theta^2)$ at the ends so that the equation reduces to

$$\int_.^t \frac{2k\sin t}{v^2}\,dt + \frac{1}{\pi}\int_0^\pi \frac{\sin\theta F'\{v(\theta)\}}{\cos t - \cos\theta}\,d\theta = 0.$$

5. Problems with restrictions. In this section a brief account will be given of the way in which the method can be modified to take account of certain restrictions peculiar to flow problems. For example, let A be the area of a symmetrical profile and D the strength of the equivalent doublet which represents the flow at large distances. As in §4

$$\delta A = \frac{1}{2}\mu^2\int_0^\pi \frac{\eta(\theta)}{\sin\theta}\left|\frac{dz}{d\zeta}\right|^2 d\theta, \tag{5.1}$$

and since δD is just the coefficient of ζ^{-1} in the perturbing potential 3.2

$$\delta D = \frac{2\mu^2}{\pi}\int_0^\pi \sin\theta\eta(\theta)\,d\theta.$$

If no restriction is made the ratio D/A is stationary only if

$$\left|\frac{dz}{d\zeta}\right|^2 = \lambda\sin^2\theta \qquad \text{or if} \qquad \left|\frac{dw}{dz}\right|$$

is constant over the whole profile. This is possible only for a strip and to avoid this trivial result, where a finite area would have to be extended indefinitely along the stream direction, it is necessary to limit the aerofoil in this direction. The simplest restriction seems to be that the aerofoil is not to lie outside of two lines drawn perpendicular to the stream. The ends then must lie on these lines and the method of variation is not applicable to such parts of the profile. Over the rest of the profile dw/dz is constant as before and the complete solution is that the aerofoils belong to the Ria-

bouchinsky constant velocity series [2]. In such cases the area must be regarded as fixed and the theory of isoperimetrical problems used to establish the constant velocity relation. For given straight lines, i.e. for a given maximum chord, the area completely determines the solution.

In the last case the restriction was geometrical but it is also possible to limit velocity instead of the chord. In the typical case there are constant velocity portions over the middle of the aerofoil and the problem is to find perturbations of the whole boundary which leave the velocity unchanged on these arcs, say the intervals $(\pi/2) \pm \lambda$ and $(3\pi/2) \pm \lambda$ for a symmetrical aerofoil. A simple method is to take a small bulge on the free arc and then a general perturbation of the constant velocity arc. In the case quoted this is achieved by way of the transformation.

$$t = \zeta + \frac{1}{\zeta} = \sin \lambda\left(s + \frac{1}{s}\right), \tag{5.4}$$

and a suitable disturbed potential is

$$w + \delta w = u\left[\zeta + \frac{1}{\zeta} + \mu^2\left\{\frac{1}{\zeta + 1/\zeta - 2\cos\chi} + \sum_1^\infty \frac{a_n s^{-n}}{\sin\lambda}\right\}\right], \tag{5.5}$$

where the a_n are to be determined by the integral equation for

$$\nu(u) \equiv \sum n a_n \sin nu$$

$$\nu(u) - \frac{\sin\lambda \sin u}{\sqrt{1 - \sin^2\lambda \cos^2 u}} \{1 + \alpha(\theta)\} \int_0^\pi K(u, t)\nu(t)\, dt = \frac{2\beta}{1 - \beta^2} \sum_1^\infty n\beta^n \sin nu \equiv G(u, \chi),$$

where

$$K \equiv \frac{2}{\pi} \sum_1^\infty \frac{\sin nu \sin nt}{n}$$

and

$$\beta = \sin\lambda/\{\cos\chi + \sqrt{\cos^2\chi - \sin^2\lambda}\}.$$

In general successive approximation would be required on account of the presence of $\alpha(\theta)$ in (5.6) but if the aerofoil is flattish or if λ is small (5.6) gives $\nu(u)$ directly to a good approximation.

The author has carried out calculations for the problem of §3 with $F \equiv v$ but requiring in addition that the maximum velocity is slightly less than twice the free stream value. In this case small constant velocity arcs appear and the method of variation gives the velocity over the rest of the profile. These solutions found by taking $\nu(u) = G(u, \chi)$ in (5.6) tend smoothly into the circle as λ vanishes. It may also be noted that the result of imposing restrictions may be to give mixed boundary conditions. For example in the preceding problem if a curve of given area π is required to lie between $y = \pm C$ where $C < 1$ the circle is replaced by a solution flattened so that $y = \pm C$ over the middle whilst the method of variation gives the velocity over the ends. This means that over one part of the aerofoil the magnitude, and over the other, the inclination of the velocity is given.

6. Minimal problems in potential theory. In dealing with aerofoils it is natural to use the conformal mapping of the region onto a circular one but this is only a device. As the following shows the true significance of the method is that it gives a functional relation between quantities on the boundary of the field. Moreover the linear transformations which arise in the method may be more compactly explained in terms of potential theory using Green's Theorem for the original and perturbing potential c.f. [3].

Let

$$I = \int_\Sigma F\left(Q, \varphi, \frac{\partial \varphi}{\partial n}\right) dS_Q', \tag{6.1}$$

where Σ is an equipotential: $\varphi = F$ and for simplicity it will be supposed that $\varphi = O(1/r)$ at great distances whilst Q expresses the dependance of F on the space-coordinates. If φ is varied the new equipotential Σ' is given by drawing from Σ a normal of length

$$\delta n = -\left[\delta\varphi \Big/ \frac{\partial \varphi}{\partial n}\right]_\Sigma. \tag{6.2}$$

Now $\delta(dS) = K\delta n\, dS$ where K is the total curvature of Σ at Q and the first variation is

$$\delta I = \int_\Sigma FK\ \delta n\, dS + \int_\Sigma F_n\ \delta n\, dS + \int_\Sigma F_{\varphi_n}\varphi_{nn}\ \delta n\, dS + \int_\Sigma F_{\varphi_n}(\delta\varphi)_n\, dS \tag{6.3}$$

suffixes denoting partial derivatives.
The last term involves $(\delta\varphi)_n$ but this may be expressed in terms of S itself by introducing a new potential Φ which takes the value of F_{φ_n} on Σ.

Then, by Green's identity,

$$\int_\Sigma F_{\varphi_n}(\delta\varphi)_n\, dS = \int \frac{\partial}{\partial n}(\Phi)\ \delta\varphi\, dS = -\int \Phi_n\varphi_n\ \delta n\, dS, \tag{6.4}$$

according to (6.2).
Applying the fundamental lemma of the calculus of variations to (6.3) the condition $\delta I = 0$ leads to

$$KF + F_n + F_{\varphi_n}\varphi_{nn} - \Phi_n\varphi_n = 0. \tag{6.5}$$

As a simple illustration let C be the capacity of an isolated conductor having a charge which gives it a potential E above that at infinity. If V is the volume $I \equiv C/V^{1/3}$ is non dimensional.
Then

$$\delta V = \int_\Sigma \delta n\, dS', \tag{6.6}$$

and for the variations just considered

$$\delta C = \frac{\delta M}{E},$$

where

$$\delta M = -\frac{1}{4\pi}\int_\Sigma \frac{\partial}{\partial n}(\delta\varphi)\, dS$$

is the change of charge on the conductor.

Hence

$$\delta C = -\frac{1}{4\pi E^2} \int_\Sigma \varphi \frac{\partial}{\partial n} (\delta\varphi)\, dS$$

$$= -\frac{1}{4\pi E^2} \int_\Sigma \frac{\partial \varphi}{\partial n} \delta\varphi\, dS$$

$$\delta C = +\frac{1}{4\pi E^2} \int_\Sigma \left(\frac{\partial \varphi}{\partial n}\right)^2 \delta n\, dS \tag{6.7}$$

From (6.6) (6.7) since δn is arbitrary the condition $\delta I = 0$ is

$$\left(\frac{\partial \varphi}{\partial n}\right)^2 = \text{const.}$$

so that the surface must be such that electricity is distributed uniformly over its surface. Thus if the solution were not known in advance this result would lead directly to the sphere as the solution.

Conclusion. The method of variation of [1] has now been presented in a form in which the author hopes it may be of practical utility. It is a direct method for aerofoils and the numerical work in solving the equations which arise is not greater than in finding approximate flows for given boundaries: in some cases much less. On the other hand it is to be expected that for smooth changes from true minimal shapes no large changes will appear. This is indeed an essential feature of all minimal solutions.

The author is indebted to Dr. W. H. J. Fuchs for explaining to him something of the beautiful work of M. Schiffer [4] on minimal problems in conformal mapping. It would appear that such problems as arise in analysis are far deeper than those needed in aerofoil theory. In the latter the condition of simple boundaries does not enter except trivially and since it is the essential condition in analysis the present method may not be of any use. It is immediately evident that the *proof* of the main lemma of the paper excludes cut regions where *inward* variations would require a second Riemann sheet.

The author also thanks Dr. P. M. Davidson for suggesting a variety of minimal problems in potential theory and in particular the capacity problem of §6 to which he originally gave this solution using instead of Green's identity for the potentials a physical interpretation of the problem together with the method of virtual work.

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WALL EFFECTS IN CAVITY FLOW—II*

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1. Introduction. In Part I of the present study,** the problems of flow about a cavitating body symmetrically placed in a channel or in a free jet have been solved in the case where the cavity extends to infinity downstream. The infinitely long cavity occurs, in each configuration, at one particular cavitation number which is a function of blockage ratio in the first case and is zero in the second. At greater values of the

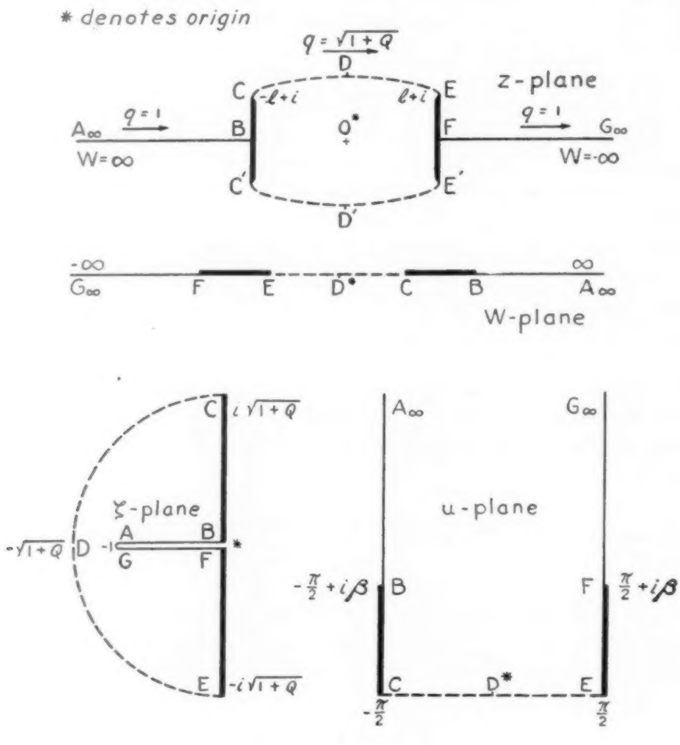

FIG. 1 - CASE A.

cavitation number, the cavity is of finite extent and a different analysis is necessary. The solutions of the corresponding problems with finite cavities are given in the present Part.

The configurations examined are again two-dimensional, this permitting the employment of conformal transformation technique. The body is taken in the form of a finite

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lamina perpendicular to the stream, so that the physical features of the flow may not be obscured by mathematical difficulties. Explicitly, the cases treated are

A. The cavitating lamina in an infinite stream;

B. The same in a channel of finite width;

C. The same in a free jet of finite width.

2. Case A. The lamina in an infinite stream. This case, where the liquid has no outer boundaries, is taken first to provide a standard for the other two. Solutions of this case have been given previously by Riabouchinsky [1] and by Fisher in an unpublished British Admiralty report, but the present treatment is much simpler than either.

Take the density of the liquid as unity, the velocity at infinity as unity, and the width of the strip forming the body as 2 units, so as to avoid unnecessary symbols. This strip is disposed between the points $(-1 \pm i)$ in the z-plane (Fig. 1). The free boundaries, there shown in broken line, start from the edges of the strip and re-form downstream on a similar, conventional, solid strip, extending between the points $(1 \pm i)$. This device, which is due to Riabouchinsky, avoids the closure jets and turbulence that would otherwise have to be taken into account. In this way, the mathematical advantages of a symmetrical problem are obtained merely by modification of the downstream conditions, to which the flow around the cavitating body is known to be insensitive. That this is so, has been clearly demonstrated by Gilbarg, Rock and Zarantonello,† who, in an as yet unpublished analysis of the similar problem with downstream closure by a re-entrant jet, find for low and moderate cavitation numbers, cavity boundaries and drag coefficients virtually indistinguishable from those that result below.

The flow being symmetrical about the x axis, consideration is restricted to the upper half z-plane. The corresponding regions in the W and ζ-planes are shown in Fig. 1, together with the auxiliary plane of u. Symbolism is as in Part I.

Proceeding by Kirchhoff's method for discontinuous flows, the transformation relations are found:

$$\zeta = -(1 + Q)^{1/2}\left[\frac{1 + i \tanh \beta \tan u}{1 - i \tanh \beta \tan u}\right]^{1/2}, \tag{1}$$

$$\frac{dz}{du} = S_1(\beta) \cos u \left[\frac{1 - i \tanh \beta \tan u}{1 + i \tanh \beta \tan u}\right]^{1/2}, \tag{2}$$

where

$$\beta = \tfrac{1}{2} \log (1 + Q) \tag{3}$$

and $S_1(\beta)$ can be evaluated in terms of standard Jacobian elliptic functions of modulus $k = \operatorname{sech} \beta$ as

$$S_1(\beta) = \frac{k^2}{k'^2 + E' - k^2 K'}. \tag{4}$$

The integration of (2) between appropriate limits then yields the cavity half-length and half-width a:

†Partial results were detailed previously by D. Gilbarg and H. H. Rock. Nav. Ord. Lab. Memo. 8718.

$$
\begin{aligned}
1 &= S_2(\beta) = \frac{E - k'^2 K}{k'^2 + E' - k^2 K'}, \\
a - 1 &= S_3(\beta) = \frac{k'(1 - k')}{k'^2 + E' - k^2 K'}.
\end{aligned} \tag{5}
$$

The intrinsic equation of the cavity boundary, referred to its point of departure C as origin is found to be

$$
s = S_1(\beta)\left[1 - \frac{\tan\theta}{(\tanh^2\beta + \tan^2\theta)^{1/2}}\right]. \tag{6}
$$

In Cartesian parametric form, this is equivalent to

$$
\begin{aligned}
x &= S_1(\beta)\left[\cosh^2\beta E\left(k, \frac{1}{2}\pi - \theta\right) - \sinh^2\beta \cdot F\left(k, \frac{1}{2}\pi - \theta\right) - \frac{\sin\theta}{(\tanh^2\beta + \tan^2\theta)^{1/2}}\right], \\
y &= S_1(\beta)\sinh^2\beta\left[\frac{\sec\theta}{(\tanh^2\beta + \tan^2\theta)^{1/2}} - 1\right],
\end{aligned} \tag{7}
$$

where E, F are the standard elliptic integrals of the second and first kinds respectively.

The drag coefficient of the lamina, based on unit velocity, is found after some reduction to be

$$
C_D = (1 + Q)S_4(\beta), \tag{8}
$$

where

$$
S_4(\beta) = \frac{2(E' - k^2K')}{k'^2 + E' - k^2K'}. \tag{9}
$$

Referred to the velocity on the cavity boundary, $(1 + Q)^{1/2}$, the drag coefficient is

$$
C_1 = S_4(\beta). \tag{10}
$$

The functions S_1, S_2, S_3, S_4 are all easily calculable and there is no difficulty in applying the foregoing solution in any numerically given case. The sub-class of cases in which Q is small is of especial interest: the functions then degenerate, giving the following simple results:

$$
\begin{aligned}
1 &= \frac{16}{\pi + 4}\left(\frac{1}{Q^2} + \frac{1}{Q}\right) + 0(1), \\
a - 1 &= \frac{8}{(\pi + 4)Q} + 0(Q), \\
C_D &= \frac{2\pi}{\pi + 4}(1 + Q) + 0(Q^2), \\
C_1 &= \frac{2\pi}{\pi + 4} + 0(Q^2).
\end{aligned} \tag{11}
$$

The cavity contour becomes

$$x = \frac{2}{\pi + 4}\left[\operatorname{cosec}\theta\cot\theta - \operatorname{gd}^{-1}\left(\frac{1}{2}\pi - \theta\right)\right] + 0(Q^2),$$
$$y = \frac{4}{\pi + 4}(\operatorname{cosec}\theta - 1) + 0(Q^2). \tag{12}$$

In the limit, as $Q \to 0$, (11) and (12) become the classical results for the lamina in an infinite stream.

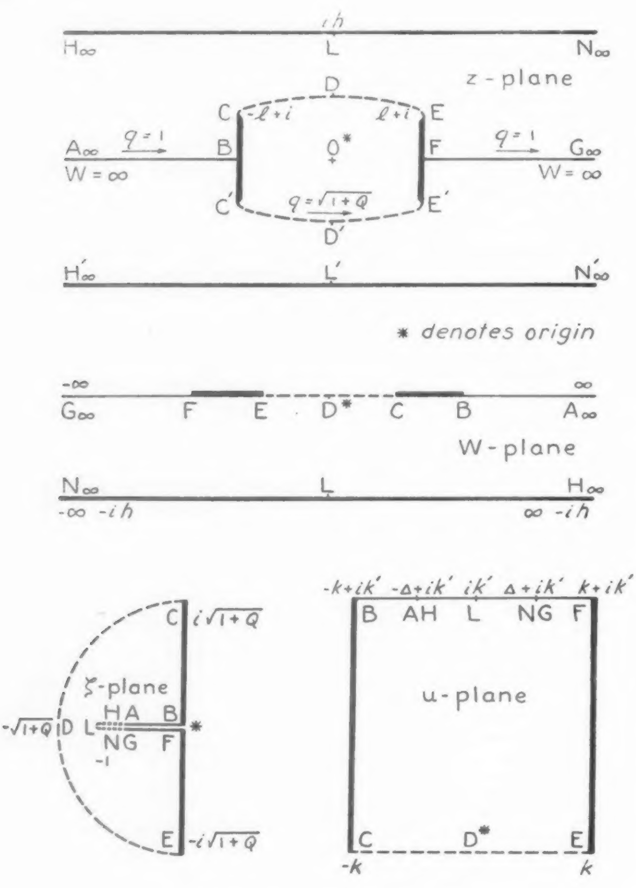

FIG. 2 - CASE B.

3. Case B. The lamina in a channel of finite width. Take the same arrangement as in Case A, but with the liquid confined between two parallel rigid walls, distant $2h$ apart, with respect to which the body is symmetrically placed (Fig. 2). Again restricting con-

sideration to the upper half z-plane, the region in the W-plane is now an infinite strip: this, together with the ζ and u-planes, is shown in Fig. 2.

Proceeding as before, but with greater complexity due to the additional singularities, one finds the following transformation equations:

$$\zeta = -(1+Q)^{1/2} \frac{\operatorname{cn} u + ik' \operatorname{sn} u}{\operatorname{dn} u}, \tag{13}$$

$$\frac{dz}{du} = \frac{2hk \operatorname{sn} \Delta}{\pi(1+Q)^{1/2}} \frac{\operatorname{cn}^2 u - ik' \operatorname{sn} u \operatorname{cn} u}{1 - k^2 \operatorname{sn}^2 \Delta \operatorname{sn}^2 u}, \tag{14}$$

where

$$\operatorname{dn} \Delta = \frac{2+Q}{Q} k'. \tag{15}$$

The constant k is not known *ab initio*, but must be determined at the end to conform with the given h.

The integration of (14), between appropriate limits, then yields after reduction the following expressions for the geometrical characteristics:

$$\frac{\pi(1+Q)^{1/2}}{2h} = \frac{dn\,\Delta}{k \operatorname{cn} \Delta} [K'E(\Delta) + (E' - K')\Delta] + \frac{k'}{k \operatorname{cn} \Delta} \cos^{-1} (\operatorname{cd} \Delta) - kK' \operatorname{sn} \Delta, \tag{16}$$

$$1 = \frac{h}{\pi(1+Q)^{1/2}} [k^2 \operatorname{sn} \Delta \ - Z(\Delta) \operatorname{dc} \Delta] \frac{2K}{k}. \tag{17}$$

$$a - 1 = \frac{2h}{\pi(1+Q)^{1/2}} \frac{k'}{k \operatorname{cn} \Delta} [\operatorname{am} \Delta - \tan^{-1} (k' \operatorname{sc} \Delta)]. \tag{18}$$

The cavity shape is obtained in Cartesian parametric form as

$$\begin{aligned} x &= \frac{h}{\pi} \frac{2+Q}{1+Q} \left[\{k^2 \operatorname{sn} \Delta \operatorname{cd} \Delta - Z(\Delta)\} v + \frac{1}{2} \log \frac{\Theta_1(\Delta + v)}{\Theta_1(\Delta - v)} \right], \\ y &= \frac{h}{\pi} \frac{Q}{1+Q} [\tan^{-1} (k' \operatorname{sc} \Delta \operatorname{nd} v) - \tan^{-1} (k' \operatorname{sc} \Delta)], \end{aligned} \tag{19}$$

where the parameter v runs from 0 to $2K$.

Again, after considerable reduction, one finds for the drag coefficients

$$\begin{aligned} C_D &= 2\left[1 + Q - \frac{hQ}{\pi} \tan^{-1} (k' \operatorname{sc} \Delta) \right], \\ C_L &= 2\left[1 - \frac{h}{\pi} \frac{Q}{1+Q} \tan^{-1} (k' \operatorname{sc} \Delta) \right]. \end{aligned} \tag{20}$$

With these relations, the solution is formally complete in its barest essentials: if the complete flow pattern is desired, the z, u relation can be found without difficulty by integration of (14) and the velocity vector at any point is then given by (13).

The numerical solution for any given case involves some complication. Given Q and h, the value of k must be found from (16); successive approximation is the indicated method. At the same time am Δ is found from (15); the remaining results can then be evaluated.

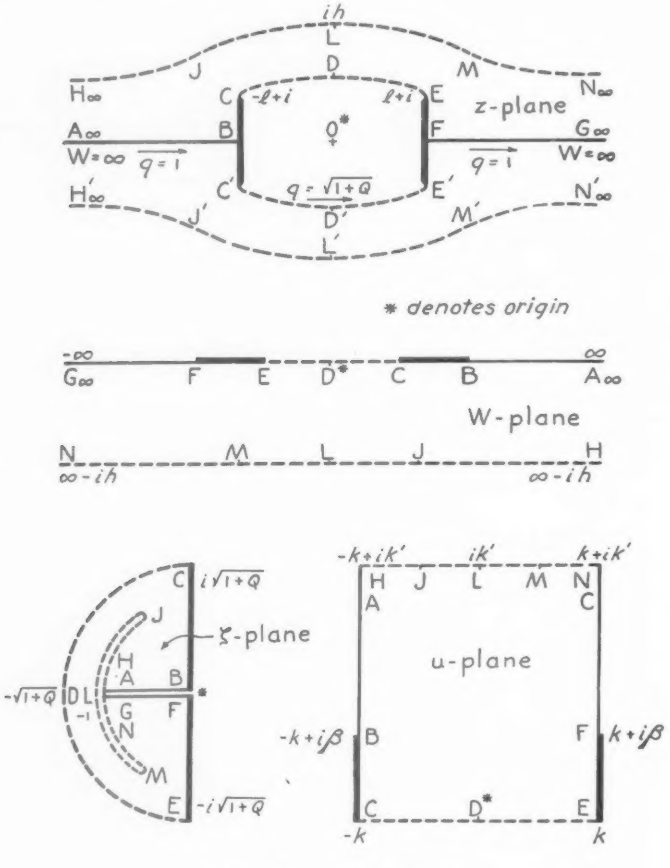

FIG. 3 - CASE C.

It is soon found, on trial, that solutions do not exist for all combinations of Q and h. For each value of Q, there is a limiting value of the blockage ratio $1/h$ that cannot be exceeded: this limiting value is given by

$$\left(\frac{1}{h}\right)_{\max} = \frac{1 + \frac{1}{2}Q - (1 + Q)^{1/2}}{1 + Q} + \frac{1}{\pi}\frac{Q}{1 + Q}\tan^{-1}\frac{Q}{2(1 + Q)^{1/2}}. \tag{21}$$

It is easily verified that, in the limiting condition, the length of the cavity is infinite, so that the solution degenerates to that of Part I. Moreover, the liquid at infinity downstream is on the point of cavitating. Hence the limitation is an inherent physical one. It bears some analogy with the choking phenomenon in a transonic wind-tunnel.

The limitation, at low cavitation numbers, is extremely harsh, e.g. at $Q = 0.05$, the blockage ratio cannot exceed about 1/1500 (cf. Part I, Sect. 2). Alternatively, for a blockage ratio of 0.05, the minimum cavitation number obtainable is 0.6.

When numerical values are considered, it is found that for admissible solutions, the drag coefficients for any given cavitation number are virtually the same as in Case A: this is due to the very low blockage. The cavity tends to be larger than with infinite fluid, i.e., in effect, the cavitation number is decreased by the fixed boundaries, especially when conditions are nearly critical, but comparison of calculated cavity contours shows that this effect becomes appreciable only at points substantially downstream.

4. Case C. The lamina in a free jet of finite width. Take the same arrangement as in Case A, but with the body symmetrically placed in a free jet whose width at infinity is $2h$ units. Still restricting consideration to the upper half z-plane, the region in the W-plane is again an infinite strip and the transformation planes of ζ and u are as shown in Fig. 3. In these planes, it is necessary to take into account the points J, M at which the boundary stream-lines inflect.

Taking account of these singularities and proceeding along the same lines as in the two previous cases, one finds the transformation relations

$$\zeta = -(1 + Q)^{1/2}\left[\frac{H_1(u - i\beta)}{H_1(u + i\beta)}\right]^{1/2}, \tag{22}$$

$$\frac{dz}{du} = \frac{2hk}{\pi(1 + Q)^{1/2}}\left[\frac{H_1(u + i\beta)}{H_1(u - i\beta)}\right]^{1/2} \operatorname{cd} u, \tag{23}$$

where

$$\beta = \frac{K}{\pi}\log(1 + Q) \tag{24}$$

and H_1 is the Jacobian theta-function constructed, like $cd\ u$, with modulus k. k in turn must be found from the complicated integral equation

$$\frac{\pi(1 + Q)^{1/2}}{2hk} = \int_0^\beta \left[\frac{H(iu + i\beta)}{H(iu - i\beta)}\right]^{1/2} \operatorname{sn} iu\, du. \tag{25}$$

In this expression, the complex radical takes its first quadrant value.

In terms of k and β, the cavity dimensions are now found to be

$$1 = \frac{hk}{\pi(1 + Q)^{1/2}}\int_0^K \frac{H(u + i\beta) + H(u - i\beta)}{\{H(u + i\beta)H(u - i\beta)\}^{1/2}} \operatorname{sn} u\, du, \tag{26}$$

$$a - 1 = \frac{hk}{\pi(1 + Q)^{1/2}}\int_0^K \frac{H(u + i\beta) - H(u - i\beta)}{i\{H(u + i\beta)H(u - i\beta)\}^{1/2}} \operatorname{sn} u\, du. \tag{27}$$

The intrinsic equation of the cavity boundary is

$$\begin{aligned} s &= \frac{2h}{\pi(1 + Q)^{1/2}}\log\frac{1 + k}{\operatorname{dn} v + k \operatorname{cn} v}, \\ \tan\theta &= \frac{H(v + i\beta) - H(v - i\beta)}{i\{H(v + i\beta) + H(v - i\beta)\}}, \end{aligned} \tag{28}$$

where the paremeter v runs from 0 to $2K$.

The drag and lift coefficients reduce to

$$C_D = \frac{2hk}{\pi}(1+Q)^{1/2}\int_0^\beta \frac{H(i\beta+iu)-H(i\beta-iu)}{\{H(i\beta+iu)H(i\beta-iu)\}^{1/2}}\frac{\operatorname{sn} iu}{i}\,du,$$

$$C_L = \frac{2hk}{\pi(1+Q)^{1/2}}\int_0^\beta \frac{H(i\beta+iu)-H(i\beta-iu)}{\{H(i\beta+iu)H(i\beta-iu)\}^{1/2}}\frac{\operatorname{sn} iu}{i}\,du. \tag{29}$$

These relations comprise the solution of Case C. It is readily shown that, as $k \to 0$, the solution degenerates to that of Case A. This however corresponds to very great values of h and is not of practical interest. In the general case, (24) and (25) must be solved simultaneously for k and β by a method of successive approximation. The remaining results can then, with some trouble, be evaluated.

The case Q small, which is of the greatest practical interest, can be approximately solved in explicit terms. For one finds that this case corresponds to $k \to 1$, so that the elliptic and theta-functions approach degenerate forms. Thus K is logarithmically large and β/K small in comparison with unity. One develops the solution in powers of Q and retains terms of order Q. Then (24) and (25) become

$$\frac{\beta}{K} = \frac{Q}{\pi},$$

$$\frac{(1+Q)^{1/2}}{h} = S_5(\beta) + \frac{Q}{\pi^2}S_6(\beta), \tag{30}$$

where

$$S_5(\beta) = 1 - \cos\beta + \frac{1}{\pi}\sin\beta\log\frac{1+\sin\beta}{1-\sin\beta},$$

$$S_6(\beta) = \pi\sin\beta\log\sec\beta + 2[f(\tan\tfrac{1}{2}\beta) - f(-\tan\tfrac{1}{2}\beta)] - \cos\beta[f(\sin\beta) - f(-\sin\beta)], \tag{31}$$

in which

$$f(x) = \sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)^2}, \tag{32}$$

a tabulated function [2]. Hence, for given values of Q and h, β is readily determined.

The simplified forms of (26), (27), (28), (29) are, respectively,

$$1 = \frac{2h}{\pi(1+Q)^{1/2}}\left[\frac{\pi}{Q}\sin\beta - \cos\beta\log\sin\beta + \log\tan\frac{1}{2}\beta\right], \tag{33}$$

$$a - 1 = \frac{2h}{\pi(1+Q)^{1/2}}\left[\frac{\pi}{Q}(1-\cos\beta) - \sin\beta\log(1+\sin\beta)\right], \tag{34}$$

$$s = \frac{2h}{\pi(1+Q)^{1/2}}\log\cosh v$$

$$(v \geq 0) \tag{35}$$

$$\theta = \cot^{-1}(\cot\beta\tanh v) - \frac{Q}{\pi}v$$

$$
\begin{aligned}
C_D &= 2h(1 + Q)^{1/2}\left[1 - \cos\beta + \frac{Q}{\pi}\sin\beta \log\sec\beta\right], \\
C_1 &= \frac{2h}{(1 + Q)^{1/2}}\left[1 - \cos\beta + \frac{Q}{\pi}\sin\beta \log\sec\beta\right].
\end{aligned}
\tag{36}
$$

The foregoing general solution for Q small is bound by the condition that β should not be small in comparison with Q. This merely implies an upper limit to the permissible width of jet and is no handicap in practice. Within the practical range of blockage ratios and cavitation numbers, the solution holds good.

When Q is very small, the following first approximations may be used:

$$
\begin{aligned}
\frac{1}{h} &= S_5(\beta), \\
1 &= \frac{2h}{Q}\sin\beta, \\
a &= \frac{2h}{Q}(1 - \cos\beta), \\
C_D &= C_1 = 2h(1 - \cos\beta).
\end{aligned}
\tag{37}
$$

When $Q \to 0$, those results become those for the infinite cavity discussed in Part I.

It is not part of the present object to give detailed numerical results for application to arbitrary configurations: these it is hoped to present elsewhere.

Acknowledgement is made to the Chief Scientist, British Ministry of Supply, for permission to publish Part II of this paper. The views expressed in the paper are those of the authors.

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A NEW VARIATIONAL PRINCIPLE FOR ISENERGETIC FLOWS*

By C. C. LIN** (*Massachusetts Institute of Technology*)

In a paper by Rubinov and the present author,[1] it is shown that the variational principle for irrotational flows of a compressible gas can be generalized to isenergetic flows. The functions to be varied are the stream function and the density distribution.

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**Consultant, U. S. Naval Ordnance Laboratory. The present work was carried out for this Laboratory and sponsored by the Office of Naval Research.

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For such flows, L. Crocco has introduced a new stream function, which depends on the entropy. The advantage of this apparent complication is to make the velocity components directly expressible in terms of the partial derivatives of this new function. Thus a single differential equation containing only this stream function can be more explicitly obtained. In this paper, it will be shown that the same integral used in earlier variational principles yields Crocco's equation when his stream function is being varied.

Following Crocco, we shall refer all speeds to the maximum speed attainable in the field. The pressure p and the density ρ are referred to suitable units consistent with this choice of typical speed. Then the isentropic acoustic speed c is

$$c^2 = \gamma \frac{p}{\rho} \tag{1}$$

for an ideal gas with ratio of specific heats γ. The condition of constant energy may be rewritten as

$$c^2 = \frac{\gamma - 1}{2} (1 - w^2), \tag{2}$$

where w is the total speed. If the stagnation density for the stream-line of zero (reference) entropy is taken as reference, the density is given by

$$\rho e^{S/R} = (1 - w^2)^{1/(\gamma-1)}, \tag{3}$$

where S is the entropy and R is the universal gas constant.

Crocco's stream function Ψ is so defined that the velocity components u and v along the directions of increasing x and y are given by

$$\begin{aligned} y^e u(1 - w^2)^{1/(\gamma-1)} &= \Psi_y , \\ y^e v(1 - w^2)^{1/(\gamma-1)} &= -\Psi_x , \end{aligned} \tag{4}$$

where $e = 0$ is the two-dimensional case and $e = 1$ is the axially symmetrical case. In the latter case, as usual, x is taken along the axis of symmetry and y is in a perpendicular direction. The vorticity is

$$\omega = v_x - u_y = y^e(1 - w^2)^{\gamma/(\gamma-1)} g(\Psi), \qquad g(\Psi) = \frac{\gamma - 1}{2\gamma R} S'(\Psi).$$

When u and v are substituted from (4), this leads to Crocco's equation for the stream function Ψ:

$$\left(1 - \frac{u^2}{c^2}\right)\Psi_{xx} - \frac{2uv}{c^2}\Psi_{xy} + \left(1 - \frac{v^2}{c^2}\right)\Psi_{yy} - e\frac{\Psi_y}{y} = y^{2e}(1 - w^2)^{(\gamma+1)/(\gamma-1)}\left(\frac{w^2}{c^2} - 1\right). \tag{5}$$

We shall now show that

$$\delta I = \delta \iint (p + \rho w^2) y^e \, dx \, dy = 0 \tag{6}$$

with suitable boundary conditions, will lead to Eq. (5). By use of (1) and (2), the integral I becomes

$$I = \frac{1}{2\gamma} \iint A(w^2) \exp\left[-\frac{2\gamma}{\gamma - 1} \int_{\Psi_0}^{\Psi} g(\Psi)\, d\Psi\right] y^{\epsilon}\, dx\, dy \tag{7}$$

with

$$A(w^2) = \{(\gamma - 1) + (\gamma + 1)w^2\}(1 - w^2)^{1/(\gamma-1)},$$

where Ψ_0 corresponds to the streamline along which $S = 0$. In this integral, w is supposed to be expressed in terms of Ψ_x and Ψ_y through (4). In particular, w^2 can be expressed in terms of $\Psi_x^2 + \Psi_y^2$ by

$$y^{2\epsilon}B(w^2) = \Psi_x^2 + \Psi_y^2, \qquad B(w^2) = w^2(1 - w^2)^{2/(\gamma-1)} \tag{8}$$

The variation δI consists of two parts: (1) the direct variation of Ψ, (2) the variation of Ψ through w^2. The latter part can be easily transformed into variation of Ψ by (8). By noting that

$$A'(w^2)/B'(w^2) = \gamma(1 - w^2)^{-1/(\gamma-1)}$$

we obtain

$$\delta I = \frac{-1}{\gamma - 1} \iint A(w^2) \exp\left[-\frac{2\gamma}{\gamma - 1} \int_{\Psi_0}^{\Psi} g(\Psi)\, d\Psi\right] g(\Psi)\, \delta\Psi\, y^{\epsilon}\, dx\, dy$$

$$+ \iint (1 - w^2)^{-1/(\gamma-1)} \{\Psi_x(\delta\Psi)_x + \Psi_y(\delta\Psi)_y\} \exp\left[-\frac{2\gamma}{\gamma - 1} \int_{\Psi_0}^{\Psi} g(\Psi)\, d\Psi\right] y^{-\epsilon}\, dx\, dy.$$

With boundary condition $\Psi_n \delta\Psi = 0$, $\delta I = 0$ leads to the equation

$$-\frac{\partial}{\partial x}\left\{(1 - w^2)^{-1/(\gamma-1)} \exp\left[-\frac{2\gamma}{\gamma - 1} \int_{\Psi_0}^{\Psi} g(\Psi)\, d\Psi\right] \frac{\Psi_x}{y^{\epsilon}}\right\}$$

$$+ \frac{\partial}{\partial y}\left\{(1 - w^2)^{-1/(\gamma-1)} \exp\left[-\frac{2\gamma}{\gamma - 1} \int_{\Psi_0}^{\Psi} g(\Psi)\, d\Psi\right] \frac{\Psi_y}{y^{\epsilon}}\right\}$$

$$= -\frac{A(w^2)}{\gamma - 1} \exp\left[-\frac{2\gamma}{\gamma - 1} \int_{\Psi_0}^{\Psi} g(\Psi)\, d\Psi\right] g(\Psi) y^{\epsilon}.$$

By direct differentiation and by use of (4) and (2), Eq. (5) is verified.

For convenience of comparison, we record that

$$\frac{d\Psi}{d\psi} = e^{S/R}$$

in the present dimensionless form, where ψ is the usual stream function.

ON MATRIX BOUNDARY VALUE PROBLEMS*

By JULIÁN ADEM AND MARCOS MOSHINSKY (*Institutos de Geofísica y Física, Universidad de México*)

Introduction. In a recent publication[1] a matrix type of boundary value problem was introduced in order to simplify the description of nuclear reactions. It appeared that this type of boundary value problem could find applications in other branches of mathematical physics, and the purpose of the present note is to illustrate them.

When we deal with vibrations of continuous media, with problems of heat flow etc., we usually describe the state of the system in terms of a single function which depends on position as well as on the time. As an example, we may mention the lateral displacement of a vibrating string, or the temperature function in case of problems of heat flow.

In many problems of vibration and heat conduction, of which examples will be given below, the description of the state by a single function leads to boundary value problems of great difficulty. It is possible though, in some cases, to divide the continuous medium into several regions, and with each region we can associate a function describing its state. These functions can be grouped together in the form of a column matrix or vector, which will then represent the state of the whole system. The mathematical problem we encounter then, is a matrix boundary value problem, which is, in general, much simpler than the one we would have to deal with in the usual formulation.

In the present note, we shall discuss two examples of matrix boundary value problems. The first one describing the flow of heat in a cross, illustrates the case where the interactions between the different regions appear through boundary conditions. The second one, dealing with the vibration of systems of plates with intermediate elastic media, illustrates the case where the interactions take place through the equations of motion. We shall obtain the eigenvalues and eigenmatrix functions corresponding to this type of problems, and with the help of them, give a formal solution for any initial conditions.

For the discussion of the self-adjoint properties of this type of boundary value problem, and the rigorous derivation of the series expansion theorems, we refer to other publications.[2,3,4]

1. Flow of heat in a cross. We shall consider the problem of flow of heat in a cross (Fig. 1a) whose four arms are of the same length l, and of square cross section of area a^2, where $a \ll l$. The material of the cross will have a density ρ, conductivity κ and specific heat c. The lateral sides of the cross will be coated in such a way that the outer conductivity[5] can be taken as zero, i.e. there is no radiation.

If we tried to deal with this problem as a three-dimensional heat conduction problem in a region bounded by the surface of the cross, we would have a difficult boundary value problem which would not admit a simple solution. Taking into account though, that the smallness of the cross section permits us to assume that the temperature at all points in it is the same, we can describe the state of temperature in the cross in the following fashion: with each bar of the cross we associate its temperature function $\theta_i(x, t)$, where $i = 1, 2, 3, 4$ indicates the bar in question, x represents the position of the point on the bar with $0 \leq x \leq l$ as indicated in Fig. 1a., and t is the time. The temperature state of the whole cross is then described by the column matrix:

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$$\boldsymbol{\theta}(x, t) = \begin{bmatrix} \theta_1(x, t) \\ \theta_2(x, t) \\ \theta_3(x, t) \\ \theta_4(x, t) \end{bmatrix}. \tag{1}$$

The equation for the temperature in each bar will be the well known equation for heat flow in rods,[5] and so the matrix representing the temperature in the cross satisfies the equation:

$$\rho c(\partial \boldsymbol{\theta}/\partial t) = \kappa(\partial^2 \boldsymbol{\theta}/\partial x^2). \tag{2}$$

The boundary conditions on the temperature at the free end points $x = 0$, can have

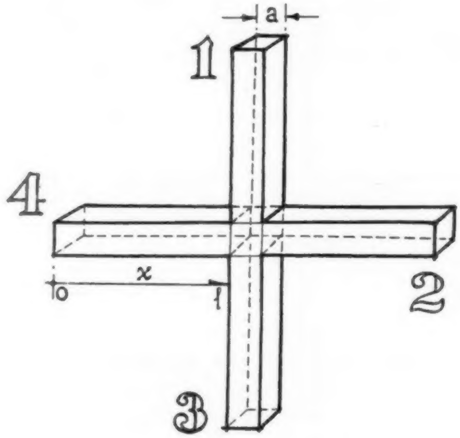

FIG. 1-a.

any of the usual forms;[5] for simplicity we will assume that the end points will be maintained at the constant temperature zero. We then have:

$$\boldsymbol{\theta}(0, t) = 0. \tag{3}$$

We now consider the boundary conditions at the end points $x = l$. The intersection of the arms of the cross at $x = l$ gives rise to a cube of linear dimensions a illustrated in Fig. 1b. The smallness of the linear dimensions of the cube, allows us to assume that the temperature at all points of the cube can be taken as the same. The temperature at the end points of all the four arms will then be equal, and we have:

$$\theta_1(l, t) = \theta_2(l, t) = \theta_3(l, t) = \theta_4(l, t), \tag{4}$$

which gives rise to three linearly independent boundary conditions.

Finally, to obtain our last boundary condition we need to consider the total flow of heat into the cube. The flow of heat per unit time through the end sections of each bar

into the cube, is given by $-\kappa a^2(\partial\theta_i/\partial x)_{x=l}$. The net inflow of heat into the cube must be equal to the increment per unit time of the quantity of heat in the cube which is $\rho c a^3(\partial\theta_1/\partial t)_{x=l}$. This quantity depends on a^3 and due to the smallness of the linear dimensions of the cube, it can be taken as of higher order than the net inflow of heat. The net flow of heat into the cube may then be assumed as ~ 0 and, as there is no radiation through the lateral sides, this leads to the boundary condition:

$$\sum_{L=1}^{4} (\partial\theta_i/\partial x)_{x=l} = 0 \tag{5}$$

The problem of flow of heat in a cross has now a complete mathematical description in terms of the equation (2) and the boundary conditions (3), (4), (5). Due to the sym-

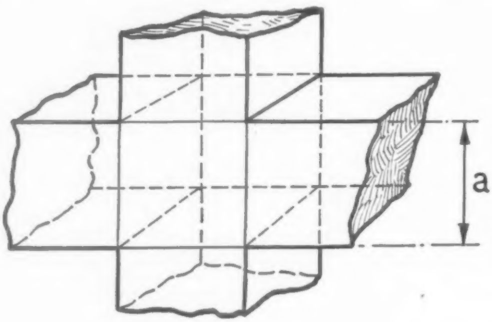

Fig. 1*b*.

metry of this particular problem, a simple linear transformation of the matrix $\boldsymbol{\theta}(x, t)$ can be found, which reduces (2)-(5) to four independent scalar problems. We shall discuss it though, as a matrix problem to illustrate the general procedure.

If we now introduce, as usual, a solution of the form $\boldsymbol{\theta}(x, t) = \boldsymbol{\theta}(x) \exp(-\lambda t)$ where λ is an arbitrary real positive constant, we are led to a matrix boundary value problem in which $\boldsymbol{\theta}(x)$ satisfies the ordinary linear equation:

$$(d^2\boldsymbol{\theta}/dx^2) + (\lambda\rho c/\kappa)\boldsymbol{\theta} = 0 \tag{6}$$

as well as the boundary conditions (3), (4), (5).

To determine the eigenvalues and eigenmatrix functions of this problem, we first notice that from (6) and (3), $\boldsymbol{\theta}(x)$ must have the form:

$$\boldsymbol{\theta}(x) = \mathbf{A} \sin (\lambda\rho c/\kappa)^{1/2}x, \tag{7}$$

where $\mathbf{A}$ is a constant column matrix of components A_i , $i = 1, 2, 3, 4$.

We now apply the boundary conditions (4), (5) to the solution (7) and we obtain the homogeneous system of linear equations:

$$\begin{gathered} A_1 \sin (\lambda\rho c/\kappa)^{1/2}l - A_i \sin (\lambda\rho c/\kappa)^{1/2}l = 0; \qquad i = 2, 3, 4 \\ \sum_{i=1}^{4} (\lambda\rho c/\kappa)^{1/2}A_i \cos (\lambda\rho c/\kappa)^{1/2}l = 0. \end{gathered} \tag{8}$$

The determinant of this system is: $4(\lambda\rho c/\kappa)^{1/2} \sin^3 (\lambda\rho c/\kappa)^{1/2}l \cos (\lambda\rho c/\kappa)^{1/2}l$, and the characteristic values for which this determinant vanishes are:

$$\lambda_n = \frac{n^2\pi^2\kappa}{4\rho c l^2}, \qquad n = 0, 1, 2, \cdots . \tag{9}$$

We see from the determinant that when n is odd, the eigenvalue is non-degenerate, while when n is even, it is triply degenerate. The corresponding eigenmatrix functions are:

for odd n:

$$\boldsymbol{\theta}_n(x) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} (2l)^{-1/2} \sin (n\pi x/2l), \tag{10a}$$

for even n:

$$\boldsymbol{\theta}_n^{(1)}(x) = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} l^{-1/2} \sin (n\pi x/2l); \qquad \boldsymbol{\theta}_n^{(2)}(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} l^{-1/2} \sin (n\pi x/2l) \tag{10b}$$

$$\boldsymbol{\theta}_n^{(3)}(x) = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} (2l)^{-1/2} \sin (n\pi x/2l).$$

We define the scalar product of two matrix functions $\boldsymbol{\phi}(x)$, $\boldsymbol{\psi}(x)$ as:

$$(\boldsymbol{\phi}, \boldsymbol{\psi}) = \int_0^l \boldsymbol{\phi}'(x)\boldsymbol{\psi}(x)\, dx = \int_0^l \left[\sum_{i=1}^4 \varphi_i(x)\psi_i(x) \right] dx \tag{11}$$

where $\boldsymbol{\phi}'(x)$ is the transposed form of the matrix $\boldsymbol{\phi}(x)$. It is then seen, that for even n, the eigenmatrix functions $\boldsymbol{\theta}_n^{(\alpha)}(x)$, $\alpha = 1, 2, 3$ were chosen so as to be mutually orthogonal, i.e. $(\boldsymbol{\theta}_n^{(1)}, \boldsymbol{\theta}_n^{(2)}) = 0$ etc. All the eigenmatrix functions are normalized $(\boldsymbol{\theta}_n, \boldsymbol{\theta}_n) = 1$. Finally, the eigenmatrix functions corresponding to different eigenvalues, are orthogonal, as can be seen directly from (10), and also from general considerations of self adjointness given in another publication.[2]

We assume now that the initial temperature distribution is given by a matrix function $\boldsymbol{\tau}(x)$ of class $C^{(1)}$, with sectionally continuous second derivative, which satisfies the

boundary conditions (3, 4, 5). The variation of temperature with time will then be represented[3,4] by a matrix function

$$\boldsymbol{\theta}(x, t) = \sum_{n=0}^{\infty} a_{2n+1}\boldsymbol{\theta}_{2n+1}(x) \exp\left[-(2n+1)^2\pi^2\kappa t/4\rho c l^2\right]$$
$$+ \sum_{n=1}^{\infty}\sum_{\alpha=1}^{3} a_{2n}^{(\alpha)}\boldsymbol{\theta}_{2n}^{(\alpha)}(x) \exp\left[-n^2\pi^2\kappa t/\rho c l^2\right], \tag{12}$$

where:

$$a_{2n+1} = (\boldsymbol{\tau}, \boldsymbol{\theta}_{2n+1}), \qquad a_{2n}^{(\alpha)} = (\boldsymbol{\tau}, \boldsymbol{\theta}_{2n}^{(\alpha)}).$$

We see that in the present formulation, the problem of flow of heat in a cross admits a complete solution.

2. Vibration of two circular plates with an intermediate elastic medium. Let us consider a system of two circular plates of radius R, clamped at the edges, and with an intermediate elastic medium. We shall designate by ρ_1 the density, D_1 the flexural rigidity and a_1, the thickness of the first plate, and ρ_2, D_2, a_2 will have the same meaning for the second plate. Finally, we denote by k the load per unit area of the plates necessary to produce a unit compression in the elastic medium.

The state of the vibrating system can then be described in terms of the normal displacements of the two plates $u_1(r, \varphi, t)$ and $u_2(r, \varphi, t)$, in which r, φ stand for polar coordinates in the plane. As the force per unit area that the plates exert on each other, is proportional to the compression $u_1 - u_2$ of the elastic medium, we have that the equations of motion[6,7] for the vibrating system are:

$$\begin{aligned} \rho_1 a_1(\partial^2 u_1/\partial t^2) + D_1\nabla^4 u_1 + k(u_1 - u_2) &= 0, \\ \rho_2 a_2(\partial^2 u_2/\partial t^2) + D_2\nabla^4 u_2 + k(u_2 - u_1) &= 0. \end{aligned} \tag{13}$$

We introduce, as usual, a solution of this system of differential equations, of the form:

$$\mathbf{u}(r, \varphi, t) = \mathbf{u}(r)\begin{Bmatrix}\cos m\varphi \\ \sin m\varphi\end{Bmatrix} \exp(i\omega t) \tag{14}$$

in which, for simplicity in notation, we abstain from associating explicitly an index m with the two components column matrix $\mathbf{u}(r)$. We are then led to the matrix boundary value problem:

$$\begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} - \frac{m^2}{r^2}\right)^2\begin{bmatrix} u_1(r) \\ u_2(r)\end{bmatrix} - \begin{bmatrix}\omega^2 a_1\rho_1 - k & k \\ k & \omega^2 a_2\rho_2 - k\end{bmatrix}\begin{bmatrix} u_1(r) \\ u_2(r)\end{bmatrix} = 0 \tag{15a}$$

$$\mathbf{u}(R) = 0 \qquad (d\mathbf{u}/dr)_{r=R} = 0 \tag{15b}$$

This type of matrix boundary value problem differs from the previous one in so far as the interactions between the components take place through the equations of motion, and not through the boundary conditions.

To find the characteristic frequencies and eigenmatrix functions of (15), we first consider the scalar boundary value problem:

$$(\mathcal{L}^2 - \lambda^2)v(r) = 0; \qquad v(R) = 0, (dv/dr)_{r=R} = 0, \tag{16}$$

where $\mathcal{L}$ is the operator $[(1/r)(d/dr)r(d/dr) - (m^2/r^2)]$, and λ is a parameter. The solution of (16) is well known, as it is the boundary value problem of a single circular plate.[7] The function $v(r)$ is given by $AJ_m(\lambda^{1/2}r) + BI_m(\lambda^{1/2}r)$ and the eigenvalues λ_n, $n = 1, 2, 3$ are given by the transcendental equation:

$$\left[J_m(\lambda^{1/2}r)\frac{d}{dr}I_m(\lambda^{1/2}r) - I_m(\lambda^{1/2}r)\frac{d}{dr}J_m(\lambda^{1/2}r)\right]_{r=R} = 0 \tag{17a}$$

The roots $\beta_n = \lambda_n^{1/2}R/\pi$ of this equation have been evaluated[7] for several values of m. The corresponding eigenfunctions are:

$$v_n(r) = A_n\left[J_m\left(\frac{\pi\beta_n r}{R}\right) - \frac{J_m(\pi\beta_n)}{I_m(\pi\beta_n)}I_m\left(\frac{\pi\beta_n r}{R}\right)\right], \tag{17b}$$

where A_n is an arbitrary constant.

We propose now a solution for (15) of the form $\mathbf{u}_n(r) = \mathbf{c}v_n(r)$ where $\mathbf{c}$ is a constant column matrix of components c_1, c_2. The boundary conditions (15b) are immediately satisfied because of (16), (17).

As $\mathcal{L}^2v_n = \lambda_n^2 v_n$, we see that the equation of motion (15a) is transformed into the algebraic linear equations

$$\left\{\begin{bmatrix} D_1\lambda_n^2 + k & -k \\ -k & D_2\lambda_n^2 + k\end{bmatrix} - \omega^2\begin{bmatrix} a_1\rho_1 & 0 \\ 0 & a_2\rho_2\end{bmatrix}\right\}\begin{bmatrix} c_1 \\ c_2\end{bmatrix} = 0 \tag{18}$$

To have non-trivial solutions of this system of equations, the determinant of the matrix must vanish, and this determines the two characteristic frequencies $\omega_n^{(1)}$, $\omega_n^{(2)}$ corresponding to each eigenvalue λ_n, which take the form

$$\left.\begin{matrix}\omega_n^{(1)} \\ \\ \omega_n^{(2)}\end{matrix}\right\} = \left\{\frac{1}{2}\left(\frac{k + D_1\lambda_n^2}{a_1\rho_1} + \frac{k + D_2\lambda_n^2}{a_2\rho_2}\right) \pm \left[\alpha^2(\lambda_n^2) + \frac{k^2}{a_1\rho_1a_2\rho_2}\right]^{1/2}\right\}^{1/2}, \tag{19}$$

where

$$\alpha(\lambda_n^2) = (2a_1\rho_1)^{-1}(k + D_1\lambda_n^2) - (2a_2\rho_2)^{-1}(k + D_2\lambda_n^2)$$

The corresponding eigenmatrices take from (18) the form

$$\mathbf{c}^{(1)} = \begin{bmatrix} k/D_1 \\ -(a_1\rho_1/D_1)\gamma(\lambda_n^2)\end{bmatrix}; \qquad \mathbf{c}^{(2)} = \begin{bmatrix} (a_2\rho_2/D_2)\gamma(\lambda_n^2) \\ k/D_2\end{bmatrix}, \tag{20}$$

where

$$\gamma(\lambda_n^2) = -\alpha(\lambda_n^2) + [\alpha^2(\lambda_n^2) + (a_1\rho_1a_2\rho_2)^{-1}k^2]^{1/2}.$$

The matrix boundary value problem (15) has now been solved, with the characteristic frequencies being $\omega_n^{(1)}$, $\omega_n^{(2)}$ and the corresponding eigenmatrix functions having the form: $\mathbf{u}_n^{(1)}(r) = \mathbf{c}^{(1)}v_n(r)$, $\mathbf{u}_n^{(2)}(r) = \mathbf{c}^{(2)}v_n(r)$.

It is easily established, from the general equations (13) and the boundary conditions (15b), that two eigenmatrix functions $\mathbf{u}^*$, $\mathbf{u}$ corresponding to different characteristic

frequencies, are orthogonal in the sense that: $\int_0^R \mathbf{u}^{*\prime}\mathbf{W}\mathbf{u}\, r dr = 0$ where $\mathbf{u}^{*\prime}$ is the transposed of $\mathbf{u}^*$ and $\mathbf{w}$ is the matrix

$$\begin{bmatrix} a_1\rho_1 & 0 \\ 0 & a_2\rho_2 \end{bmatrix}.$$

We can normalize the eigenmatrix function $\mathbf{u}$ in the sense that $\int_0^R \mathbf{u}'\mathbf{W}\mathbf{u}\, r dr = 1$ by choosing the constant in (17b) appropriately. We would then have

$$(\mathbf{u}_n^{(\alpha)}, \mathbf{u}_l^{(\beta)}) \equiv \int_0^R \mathbf{u}_n^{(\alpha)\prime}\mathbf{W}\mathbf{u}_l^{(\beta)} r\, dr = \delta_{\alpha\beta}\delta_{nl}\,, \tag{21}$$

where α, $\beta = 1, 2$, and n, $l = 1, 2, 3, \cdots$.

We have obtained an orthonormalized set of eigenmatrix functions corresponding to this vibration problem. With their help, we could represent the state of vibration for the two plates corresponding to any initial conditions. For example, let us assume that we had at $t = 0$ a given displacement for our two plates, and that the initial velocity was zero. The initial displacement of the two plates could be represented as sum of terms of the form

$$\boldsymbol{\tau}(r)\begin{Bmatrix} \cos m\varphi \\ \sin m\varphi \end{Bmatrix} \qquad \text{for} \qquad m = 0, 1, 2, \cdots.$$

For each term of this type, the solution of the vibration problem would be

$$\mathbf{u}(r, t)\begin{Bmatrix} \cos m\varphi \\ \sin m\varphi \end{Bmatrix}$$

where $\mathbf{u}(r, t)$ has the form:

$$\mathbf{u}(r, t) = \sum_{n=1}^{\infty} \{(\boldsymbol{\tau}, \mathbf{u}_n^{(1)})\mathbf{u}_n^{(1)}(r) \cos \omega_n^{(1)} t + (\boldsymbol{\tau}, \mathbf{u}_n^{(2)})\mathbf{u}_n^{(2)}(r) \cos \omega_n^{(2)} t\} \tag{22}$$

and

$$(\boldsymbol{\tau}, \mathbf{u}_n^{(\alpha)}) \equiv \int_0^R \boldsymbol{\tau}'(r)\mathbf{W}\mathbf{u}_n^{(\alpha)}(r) r\, dr.$$

The generalization of the present developments to systems of more than two plates, as well as to other types of boundary conditions, and other forms for the plates, is straightforward.

3. Conclusion. A general type of matrix boundary value problem, which includes both of the preceding cases, would be the one in which there are interactions between the components of the column matrix, both, in the differential equations and in the boundary conditions. If we introduce additional components into the column matrix, so as to reduce the system of differential equations to one of the first order, the matrix boundary value problems introduced in the present note, reduce to a form which has been extensively discussed by Birkhoff and Langer.[3] While, in their formulation, the self-adjoint nature of the present problems is obscured, their proofs concerning the existence and properties of eigenvalues and eigenmatrix functions, as well as of expansion

theorems, apply to the problems discussed above. We are justified then in using formal expansion theorems, such as (12), (22) in the solution of problems of the above type.

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A NOTE ON A VECTOR FORMULA*

By H. LOTTRUP KNUDSEN (*The Royal Technical University of Denmark, Copenhagen*)

Of some vector formulas compiled in a recent paper[1] the one discussed in the present note seems to be of general interest in field theory.

1. Derivation of the vector formula. Let $\mathbf{B}(\mathbf{r})$ denote a vector function of the position vector $\mathbf{r}$, satisfying sufficient continuity and differentiability conditions, and let A denote a closed surface and V the region of space bounded by this surface. Using conventional vector notation we may then state Gauss' theorem in the following way

$$\int_A d\mathbf{a}\cdot\mathbf{B} = \int_V dv\, \nabla\cdot\mathbf{B}. \tag{1}$$

Letting $\varphi(\mathbf{r})$ denote a scalar function and $\mathbf{\Phi}(\mathbf{r})$ a dyade function, both possessing sufficient continuity and differentiability properties, we may derive the following equations from Gauss' theorem

$$\int_A d\mathbf{a}\,\varphi = \int_V dv\, \nabla\varphi, \tag{2}$$

$$\int_A d\mathbf{a}\cdot\mathbf{\Phi} = \int_V dv\, \nabla\cdot\mathbf{\Phi}.\,[2] \tag{3}$$

Substituting in equations (2) and (3)

$$\varphi = \mathbf{r}\cdot\mathbf{B}, \tag{4}$$

$$\mathbf{\Phi} = \mathbf{r}\mathbf{B}, \tag{5}$$

*Received May 9, 1951.

[1] H. L. Knudsen, *Nogle vektorformler og deres anvendelse*, (Some vector formulas and their application), Fysisk Tidsskrift, Copenhagen, to be published.

[2] M. Lagally, *Vorlesungen über Vektor-Rechnung*, dritte Auflage, Leipzig, 1944.

and using

$$\nabla \cdot \mathbf{r} = 3, \tag{6}$$

$$\nabla \mathbf{r} = \boldsymbol{\varepsilon}, \tag{7}$$

where $\boldsymbol{\varepsilon}$ is the unit dyade, we obtain

$$\begin{aligned}\int_A d\mathbf{a}\, \mathbf{r}\cdot\mathbf{B} &= \int_V dv\, \nabla(\mathbf{r}\cdot\mathbf{B}) = \int_V dv\, [\nabla\mathbf{r}\cdot\mathbf{B} + \nabla\mathbf{B}\cdot\mathbf{r}] \\ &= \int_V dv\, [\boldsymbol{\varepsilon}\cdot\mathbf{B} + \nabla\mathbf{B}\cdot\mathbf{r}] = \int_V dv\, [\mathbf{B} + \nabla\mathbf{B}\cdot\mathbf{r}],\end{aligned} \tag{8}$$

$$\int_A d\mathbf{a}\cdot\mathbf{r}\mathbf{B} = \int_V dv\, \nabla\cdot(\mathbf{r}\mathbf{B}) = \int_V dv\, [\nabla\cdot\mathbf{r}\mathbf{B} + \mathbf{r}\cdot\nabla\mathbf{B}] = \int_V dv\, [3\mathbf{B} + \mathbf{r}\cdot\nabla\mathbf{B}]. \tag{9}$$

Subtracting equation (8) from (9) we find

$$\int_A d\mathbf{a}\cdot\mathbf{r}\mathbf{B} - \int_A d\mathbf{a}\, \mathbf{B}\cdot\mathbf{r} = \int_V dv\, [2\mathbf{B} + \mathbf{r}\cdot\nabla\mathbf{B} - \nabla\mathbf{B}\cdot\mathbf{r}] \tag{10}$$

or introducing the unit dyade $\boldsymbol{\varepsilon}$

$$\int_A d\mathbf{a}\cdot[\mathbf{r}\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}\mathbf{r}]\cdot\mathbf{B} = \int_V dv\, [2\mathbf{B} + \mathbf{r}\cdot\nabla\mathbf{B} - \nabla\mathbf{B}\cdot\mathbf{r}]. \tag{11}$$

We consider now the special case where $\mathbf{B}(\mathbf{r})$ is an irrotational field, i.e. we assume that

$$\nabla \times \mathbf{B} = 0. \tag{12}$$

When this condition is satisfied, the dyade $\nabla\mathbf{B}$ is symmetrical. We have then

$$\mathbf{r}\cdot\nabla\,\mathbf{B} = \nabla\mathbf{B}\cdot\mathbf{r}. \tag{13}$$

Introducing this equation in (11) we finally obtain the following equation for an irrotational field $\mathbf{B}(\mathbf{r})$

$$\frac{1}{2}\int_A d\mathbf{a}\cdot[\mathbf{r}\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}\mathbf{r}]\cdot\mathbf{B} = \int_V dv\, \mathbf{B}. \tag{14}$$

By the theorem expressed through this equation the volume integral of an irrotational vector field over a region of space may be converted into a surface integral extended over the surface bounding this region. That this conversion can be carried out, is evident from the fact that an irrotational vector field $\mathbf{B}(\mathbf{r})$ may be expressed as the gradient of a certain scalar field, the potential $\varphi(\mathbf{r})$,

$$\mathbf{B} = \nabla\varphi. \tag{15}$$

The conversion of the volume integral into a surface integral follows then directly from (2). However, the theorem expressed by equation (14) has the advantage that by using this equation we may express the surface integral without knowing the potential $\varphi(\mathbf{r})$ of $\mathbf{B}(\mathbf{r})$.

The use of the theorem developed in this section will be illustrated by an example.

2. The force on a body in a central field of force. Let a body V, bounded by a closed surface A, be acted on by a central field of force. The center of the field is denoted by O and the force per unit volume of the body by $\mathbf{f}(\mathbf{r})$, where $\mathbf{r}$ is the position vector with O as origo. The force density $\mathbf{f}(\mathbf{r})$ may be expressed as

$$\mathbf{f}(\mathbf{r}) = g(|\,\mathbf{r}\,|)\boldsymbol{\varrho}, \tag{16}$$

where $g(|\,\mathbf{r}\,|)$ is a scalar function of the distance from the center of the field, and where $\boldsymbol{\varrho}$ denotes a unit vector coparallel with $\mathbf{r}$.

The force $\mathbf{F}$, with which the field of force acts on the body V, is expressed by

$$\mathbf{F} = \int_V \mathbf{f}\, dv. \tag{17}$$

A central field of force being irrotational, we may convert the volume integral in this expression into a surface integral, extended over the boundary A of V by using the equation (14), derived in the last section. We find

$$\mathbf{F} = \frac{1}{2}\int_A d\mathbf{a}\cdot[\mathbf{r}\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}\mathbf{r}]\cdot\mathbf{f}. \tag{18}$$

Letting $\mathbf{n}$ denote the outward unit normal to A and denoting the scalar surface element by da, so that $d\mathbf{a} = \mathbf{n}\, da$, we may rewrite (18) as

$$\mathbf{F} = \frac{1}{2}\int_A [\boldsymbol{\varrho}\cos(\mathbf{n}, \boldsymbol{\varrho}) - \mathbf{n}]\,|\,\mathbf{r}\,|\,g(|\,\mathbf{r}\,|)\, da. \tag{19}$$

The expression in the square bracket in this equation has the simple geometrical meaning demonstrated in Fig. 1.

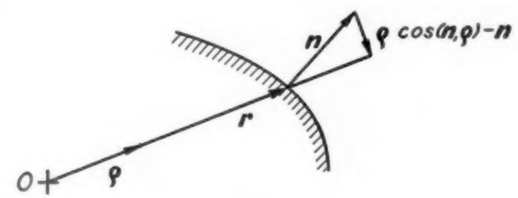

FIG. 1.

The application of the above developed expression (19) for the force on a body in a central field of force will be illustrated by two examples.

The force inversely proportional to the distance. Let us first consider a central field of force, in which the force is directed towards the center and is inversely proportional to the distance from the center. For such a field of force the function $g(|\,\mathbf{r}\,|)$ is expressed by

$$g(|\,\mathbf{r}\,|) = -K\,|\,\mathbf{r}\,|^{-1} \tag{20}$$

where K is a constant. By substituting this function in (19) we obtain the following expression for the force on the body in question

$$\mathbf{F} = -\frac{K}{2}\int_A [\boldsymbol{\varrho}\cos(\mathbf{n}, \boldsymbol{\varrho}) - \mathbf{n}]\, da = -\frac{K}{2}\int_A \boldsymbol{\varrho}\cos(\mathbf{n}, \boldsymbol{\varrho})\, da \tag{21}$$

as we have

$$\int_A \mathbf{n}\, da = 0, \tag{22}$$

this integral expressing the vector areal of the closed surface A.

As an example of the application of the formula (21) we shall calculate the force on a sphere with center P and radius R in a central field of force of the type discussed here, supposing that the center O of the field is situated on the surface of the sphere as shown in Fig. 2. From the symmetry it follows that the resulting force will be parallel

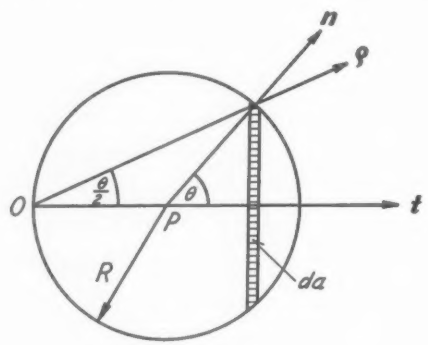

FIG. 2.

with the radius PO to the center of the field; in computing the force we therefore only need to retain the component in this direction of each of the differential contributions to the surface integral. Using the symbols shown in Fig. 2 we find from (21) the following expression for the force $\mathbf{F}$

$$\mathbf{F} = -\frac{K}{2}\int_0^\pi \mathbf{t} \cos\frac{\theta}{2} \cos\frac{\theta}{2}\, 2\pi R \sin\theta\, R\, d\theta = -K\pi R^2 \mathbf{t}, \tag{23}$$

where the first factor $\cos \theta/2$ stands for $\cos(\boldsymbol{\varrho}, \mathbf{t})$, the second factor $\cos\theta/2$ for $\cos(\mathbf{n}, \boldsymbol{\varrho})$, and $2\pi R \sin\theta\, Rd\theta$ for da. In this equation $\mathbf{t}$ denotes a unit vector coparallel with OP. Through direct calaculation of the volume integral (17) we find, by dividing the sphere in conical shells with their apex at the center O of the field and introducing the angle $\alpha = \theta/2$ as an integration variable,

$$\mathbf{F} = -\int_0^{\pi/2}\int_0^{2R\cos\alpha} \mathbf{t}K\xi^{-1} \cos\alpha\, \xi\, d\alpha\, 2\pi\xi \sin\alpha\, d\xi = -K\pi R^2\mathbf{t}, \tag{24}$$

where $\cos\alpha$ stands for $\cos(\boldsymbol{\varrho}, \mathbf{t})$, $\xi d\alpha\, 2\pi\xi \sin\alpha\, d\xi$ for dv.
In the example discussed here a double integral has to be calculated for finding the force as a volume integral, whereas by using the vector formula (14) we get the force expressed as a single integral.

The Force Inversely Proportional to the Square of the Distance. For a central field of force in which the force is inversely proportional to the square of the distance from the center

of the field and directed towards this center, the scalar function $g(|\mathbf{r}|)$ defined in (15) is expressed by

$$g(|\mathbf{r}|) = -K|\mathbf{r}|^{-2}, \tag{25}$$

where K is a constant. The force $\mathbf{F}$, with which this field acts on a body V, bounded by a closed surface A, may be found by substituting (25) in (19). We hereby find

$$\mathbf{F} = -\frac{K}{2}\int_A [\boldsymbol{\varrho}\cos(\mathbf{n}, \boldsymbol{\varrho}) - \mathbf{n}]\,|\mathbf{r}|^{-1}\,da. \tag{26}$$

For demonstrating the application of this formula we consider a sphere with center P and radius R in a central field of force of the type discussed here. The center O of the field is assumed to be situated on the surface of the sphere. The previously used Fig. 2 may also be applied as an illustration of the present problem. For reasons of symmetry the force will be parallel with the radius PO to the center of the field; in computing the surface integral we therefore retain only the component in this direction of the differential contributions. We find from (26)

$$\mathbf{F} = -\frac{K}{2}\int_0^\pi \mathbf{t}\left[\cos\frac{\theta}{2}\cos\frac{\theta}{2} - \cos\theta\right]\left[2R\cos\frac{\theta}{2}\right]^{-1} 2\pi R\sin\theta\, R\,d\theta = -\frac{4\pi}{3}KR\mathbf{t}, \tag{27}$$

where the first factor $\cos\theta/2$ stands for $\cos(\boldsymbol{\varrho}, \mathbf{t})$, the second factor $\cos\theta/2$ for $\cos(\mathbf{n}, \boldsymbol{\varrho})$, $\cos\theta$ for $\cos(\mathbf{n}, \mathbf{t})$, $[2R\cos\theta/2]^{-1}$ for $|\mathbf{r}|^{-1}$ and $2\pi R\sin\theta\, Rd\theta$ for da.

From the theory for Newtonian potentials[3] it is known that the force $\mathbf{F}$ from the field of force discussed here may be computed by assuming that the mass of the sphere is concentrated to its center. As the volume of the sphere is $4\pi/3\ R^3$ and the distance between the center of the sphere and the center of the field R, we therefore find the force $\mathbf{F}$ expressed as

$$\mathbf{F} = -(KR^{-2})\left(\frac{4\pi}{3}R^3\right)\mathbf{t}. \tag{28}$$

This expression is seen to be identical with the expression (27), found by using the surface integral method. The result from the potential theory used above was derived by carrying out a double integration. The calculation has consequently been simplified also in the present case by the use of the vector formula (14).

[3] O. D. Kellogg, Foundations of potential theory, Berlin, 1929.

THE CIRCULAR PLATE WITH ECCENTRIC HOLE*

By SAMUEL D. CONTE (*Wayne University*)

1. Introduction. It is well known that the equation governing the small deflections of a thin uniform plate, assumed homogeneous and isotropic, is

$$D\Delta\Delta w + P(x, y) = 0, \tag{1}$$

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where $P(x, y)$ is an arbitrary load per unit area acting normal to the upper surface, $D = 2d^3/3(1 - \sigma^2)$ is the flexural rigidity of the plate, d is the thickness of the plate, and Δ the Laplacian operator in two dimensions. All quantities appearing in this and in all succeeding equations are in dimensionless form. It is assumed that there are no forces acting on the edges of the plate other than the reactions due to the method of fixing of the plate. The state of stress and strain at every point of the plate can be completely determined when the deflection w of points originally on the middle plane of the plate is known. Complete solutions of the plate equation (1) under arbitrary normal loads and for various boundary conditions are known for thin rectangular [1], circular [2], or elliptic [3] plates. It is proposed in this paper to present a solution of this problem when the plate is bounded by two circles, one of which is interior to and eccentric to the other.

2. Bipolar coordinates. The boundary conditions are considerably simplified when bipolar coordinates (α, β) are introduced. The conformal transformation

$$z = x + iy = c \tanh \tfrac{1}{2} (\alpha + i\beta) = c \tanh \tfrac{1}{2}\zeta, \tag{2}$$

maps the rectangular region $[\alpha_1 \leqq \alpha \leqq a_0 , -\pi \leqq \beta \leqq \pi]$ in the ζ-plane onto the region between two eccentric circles in the z-plane. As shown in Fig. 1, the boundary of this region consists of the exterior circle $\alpha = \alpha_1$ and the interior circle $\alpha = \alpha_0$, with $\alpha_0 > \alpha_1$.

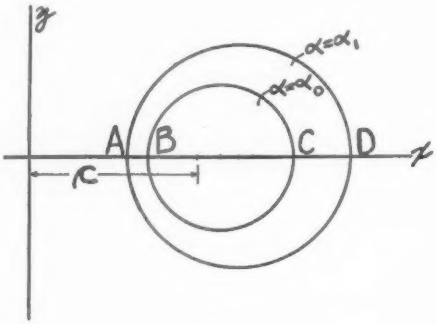

FIG. 1.

The constant c is the pole of the system. The straight line segments AB and CD are given respectively by $\beta = 0$ and by $\beta = \pi$. Under the transformation (2) the Laplacian operator becomes

$$\Delta = h^2\left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}\right), \tag{3}$$

where $(ch)^2 = (\cosh \alpha + \cos \beta)^2$ is the Jacobian of the transformation. If (hw) is taken as the dependent variable instead of w, the harmonic and the biharmonic equations become respectively,

$$\Delta w = \left\{ch \left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}\right) - 2 \sinh \alpha \frac{\partial}{\partial\alpha} + 2 \sin \beta \frac{\partial}{\partial\beta} + \cosh \alpha - \cos \beta\right\}(hw) = 0,$$

$$\Delta\Delta w = h^3\left(\frac{\partial^4}{\partial\alpha^4} + \frac{2\partial^4}{\partial\alpha^2\, \partial\beta^2} + \frac{\partial^4}{\partial\beta^4} - 2 \frac{\partial^2}{\partial\alpha^2} + 2 \frac{\partial^2}{\partial\beta^2} + 1\right)(hw) = 0.$$

Thus the biharmonic equation becomes a linear partial differential equation with constant coefficients [4]. Separation of variables now yields as solutions the following series of biharmonic functions which have the period 2π in β;

$$hw = \sum_{n=0}^{\infty} \phi_n(\alpha) \cos n\beta + \psi_n(\alpha) \sin n\beta, \tag{4}$$

where
$$\phi_0(\alpha) = A_0 \sinh \alpha + B_0\alpha \sinh \alpha + D_0 \cosh \alpha + E_0\alpha \cosh \alpha,$$

$$\psi_0(\alpha) = 0,$$

$$\phi_1(\alpha) = A_1 + B_1\alpha + D_1 \cosh 2\alpha + E_1 \sinh 2\alpha,$$

$$\psi_1(\alpha) = A_1' + B_1'\alpha + D_1' \cosh 2\alpha + E_1' \sinh 2\alpha,$$

$$\phi_n(\alpha) = A_n \cosh (n+1)\alpha + B_n \cosh (n-1)\alpha + D_n \sinh (n+1)\alpha + E_n \sinh (n-1)\alpha,$$

$$\psi_n(\alpha) = A_n' \cosh (n+1)\alpha + B_n' \cosh (n-1)\alpha + D_n' \sinh (n+1)\alpha + E_n' \sinh (n-1)\alpha,$$

and $A_n, B_n, D_n, E_n, A_n', B_n', D_n', E_n'$ $(n = 0, 1, \cdots)$ are eight sets of arbitrary constants.

3. **Method of solution.** It is now proposed to obtain a solution of the boundary value problem consisting of Eq. (1) and one of the usual boundary conditions. The most common of these boundary conditions are those corresponding to clamped edges and to simply supported edges. For a clamped edge the boundary conditions in terms of (hw) are

$$hw = \frac{\partial(hw)}{\partial\alpha} = 0. \tag{5}$$

For a simply supported edge the boundary conditions are

$$(hw) = 0,$$

$$\left\{\cosh \alpha\left(\frac{\partial^2}{\partial\alpha^2} + \sigma \frac{\partial^2}{\partial\beta^2}\right) - (1+\sigma) \sinh \alpha \frac{\partial}{\partial\alpha} + (1+\sigma) \sin \beta \frac{\partial}{\partial\beta} + \sigma \cosh \alpha - \cos \beta\right\}(hw) = 0. \tag{6}$$

A solution of the boundary value problem consisting of Eq. (1) and conditions (5) or (6) is sought in the form

$$hw = w_0 + w_1 , \tag{7}$$

where w_0 is the series of biharmonic functions (4) and w_1/h is a particular solution of the plate equation. If w_1 is expanded into a Fourier series, boundary conditions (5) or (6) give rise to a system of eight equations which can be solved for the eight sets of unknowns

which appear in w_0. If, for example, a clamped plate is subjected to a uniform normal load p, a particular solution of the resulting boundary value problem is

$$\frac{w_1}{h} = \frac{A}{h^2}, \qquad A = -\frac{pc^2}{16D}, \tag{8}$$

and since the deflection must be an even function of β, the biharmonic function is taken as

$$w_0 = \sum_{n=0}^{\infty} \phi_n(\alpha) \cos n\beta. \tag{9}$$

If w_1 is expanded into a Fourier series, the boundary conditions yield four sets of equations for the coefficients A_n, B_n, D_n, E_n which appear in $\phi_n(\alpha)$. Explicit expressions for these coefficients are presented in the author's thesis, and it is shown there that the series (9) with these coefficients converges absolutely and uniformly in α and β over the domain between the eccentric circles.

4. Particular solution for an arbitrary analytic load. Consider first a load function which is homogeneous of degree n in x and y. Expressed in bipolar coordinates such a load has the form

$$P(\alpha, \beta) = \sum_{k=0}^{n} \mathfrak{A}_{nk} \frac{\sinh^{n-k} \alpha \sin^k \beta}{h^n}. \tag{10}$$

A particular solution of the plate equation (1) for this load function is given by

$$\frac{w_1}{h} = \sum_{k=0}^{n} A_{nk} \frac{\sinh^{n-k+2} \alpha \sin^{k+2} \beta}{h^{n+4}}, \tag{11}$$

where the coefficients A_{nk} are determined by the following system of equations

$$\begin{aligned} &2(n-k+2)(n-k+1)(k+2)(k+1)A_{nk} \\ &\quad + (n-k+4)(n-k+3)(n-k+2)(n-k+1)A_{n(k-2)} \\ &\quad + (k+4)(k+3)(k+2)(k+1)A_{n(k+2)} = \mathfrak{A}_{nk}, \qquad (k = 0, 1, \cdots, n). \end{aligned} \tag{12}$$

This system of $(n + 1)$ equations can be solved uniquely for the $(n + 1)$ unknowns A_{nk} $(k = 0, 1, \cdots, n)$. These results for $k = 2m$, $k = 2m + 1$ are:

$$\begin{aligned} A_{n(2m)} = &\sum_{r=0}^{m-1} (-1)^r \frac{(n-2m+2r+2) \cdots (n-2m+3)(r+1)}{(2m+2) \cdots (2m-2r-1)} \\ &\qquad \cdot \left\{1 - \frac{(m+1)(r+[n/2]-m+2)}{(r+1)([n/2]+2)}\right\} \mathfrak{A}_{n(2m-2r-2)} \\ &+ \sum_{r=0}^{[n/2]-m} (-1)^{r+m+[n/2]} \frac{(n+2) \cdots (n-2m+3)}{(2[n/2]+4)(2m+1)!} \\ &\qquad \cdot \frac{(r+1)(2r+\epsilon_1^n)}{(n+2) \cdots (2[n/2]-2r+1)} \cdot \mathfrak{A}_{n(2[n/2]-2r)}, \end{aligned} \tag{13}$$

$$
\begin{aligned}
A_{n(2m+1)} = \sum_{r=0}^{[(n+1)/2-1]} & (-1)^r \frac{(n - 2m + 2r + 1) \cdots (n - 2m + 2)(r + 1)}{(2m + 3) \cdots (2n - 2r)} \\
& \cdot \left\{1 - \frac{(m + 1)(r + [(n + 1)/2] - m + 1)}{([(n + 1)/2] + 1)(r + 1)}\right\} \mathcal{A}_{n(2m-2r-1)} \\
+ \sum_{r=0}^{[n/2]-m+\epsilon_2^n} & (-1)^{r+m+[n/2]-\epsilon_2^n} \frac{(n + 1) \cdots (n - 2m + 2)(m + 1)}{([(n + 1)/2] + 1)(2m + 3)!} \\
& \cdot \frac{(r + 1)(2r + \epsilon_2^n)!}{(n + 1) \cdots (2[(n + 1)/2] - 2r)} \cdot \mathcal{A}_{n(2[(n+1)/2]-2r-1)}.
\end{aligned}
\tag{14}
$$

In these equations $[n/2]$, for example, means the largest integer in $n/2$, and the symbol ϵ_m^n has the following meaning: $\epsilon_m^n = 0$ if $(n + m)$ is odd; $\epsilon_m^n = 1$ if $(n + m)$ is even. All factors in these equations must appear in descending order. When this is not the case, as for example, when $r = 0$, the entire factor is to be replaced by unity.

Consider now an analytic load function $P(x, y)$. If its Taylor's series is rearranged in homogeneous powers of degree n in x and y together and expressed in bipolar coordinates, it has the form

$$
P(\alpha, \beta) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \mathcal{A}_{nk} \frac{\sinh^{n-k} \alpha \sin^k \beta}{h^n} \tag{15}
$$

where $\mathcal{A}_{nk}$ are known constants. By means of the method shown above a particular solution of the plate equation (1) is given formally by

$$
\frac{w_1}{h} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A_{nk} \frac{\sinh^{n-k+2} \alpha \sin^{k+2} \beta}{h^{n+4}} . \tag{16}
$$

The coefficients A_{nk} for each n and k can be determined by means of Eqs. (13), (14).

5. Special loads. When the load function has certain specialized forms, the peculiar properties of the operator Δ in bipolar coordinates may be profitably exploited to yield comparatively simple solutions of the boundary value problems in thin plate theory. It may be proved by induction or by direct calculation that if $u_n = \sinh n\alpha/h^n$, or if $u_n = \cosh n\alpha/h^n$, then

$$
\Delta u_n = \frac{2n^2}{c} u_{n-1} ;
$$

and if $v_n = \cos n\beta/h^n$ or if $v_n = \sin n\beta/h^n$, then

$$
\Delta v_n = \frac{-2n^2}{c} v_{n-1} .
$$

Thus, if the load function is any finite or infinite sum of the functions u_n or v_n, this stepping-down property of the operator Δ may be utilized to obtain a particular solution of the plate equation. A complete solution satisfying boundary conditions (5) or (6) may then be obtained by following the procedure given in Section 3.

As another special case let the load function be

$$
P(\alpha, \beta) = kh^3 \sinh p\alpha \cos q\beta, \tag{17}
$$

where k is a known constant and p, q are integers but $p \neq (q \pm 1)$. If the plate is clamped at the edges, a particular solution of (1) is readily obtained in the form

$$w_1 = A \sinh p\alpha \cos q\beta,$$

where

$$A = -\frac{k}{D} \frac{1}{(p^2 - q^2)^2 - 2q^2 - 2p^2 + 1}.$$

The boundary conditions may be satisfied by taking only those biharmonic terms involving $\cos q\beta$. Hence a complete closed form solution for a clamped plate under the load (17) is given by

$$hw = A \sinh p\alpha \cos q\beta + \phi_q(\alpha) \cos q\beta,$$

where $\phi_q = A_q \cosh (q + 1)\alpha + B_q \cosh (q - 1)\alpha + D_q \sinh (q + 1)\alpha + E_q \sinh (q - 1)\alpha$. The boundary conditions (5) yield four equations which determine uniquely the four constants A_q, B_q, D_q, E_q. Indeed, a solution in closed form can always be obtained if the normal load is any finite sum of terms of the following forms: $h^3 \cosh p\alpha \cos q\beta$, $h^3 \cosh p\alpha \sin q\beta$, $h^3 \sinh p\alpha \cos q\beta$, $h^3 \sinh p\alpha \sin q\beta$. If $p = (q \pm 1)$ in any of these loads the function is itself biharmonic. If, for example,

$$P(\alpha, \beta) = k\, h^3 \sinh (q + 1)\alpha \sin q\beta, \tag{18}$$

a particular solution of the plate equation which is periodic in β is given by

$$w_1 = A\, \alpha \cosh (q + 1)\alpha \sin q\beta$$

with

$$A = -\frac{k}{D} \cdot \frac{1}{2q(q + 1)(q + 4)}.$$

A closed form solution involving only four of the biharmonic functions can again be obtained as in the previous case. We note, however, that if the plate is simply supported at the edges, no finite number of biharmonic terms will suffice to yield a solution of the resulting boundary value problem under loads of the form (17) or (18).

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AN EXPANSION OF FREQUENCY DETERMINANTS WITH APPLICATION TO THE NORMAL FREQUENCIES OF A SPRING MOUNTED RIGID BODY (RESILIENT FOUNDATION)*

By F. K. G. ODQVIST (*Royal Institute of Technology, Stockholm*)

1. Introduction. The free small oscillations of a conservative system of n masses Q_i with the coordinates q_i and velocities $\dot{q}_i$ are governed by Lagrange's equations of the type

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial W}{\partial q_i} = 0, \qquad i = 1, \cdots, n, \tag{1.1}$$

where the kinetic energy is

$$T = \tfrac{1}{2} \sum_i Q_i \dot{q}_i^2 ,$$

and the potential energy is

$$W = \tfrac{1}{2} \sum_{i,j} k_{ij} q_i q_j .$$

In the sequel we shall limit the treatment to the case where the masses Q_i and the spring constants k_{ij} are constants in space and time.

The solution of (1.1) may be put in the form

$$q_i = A_i e^{i\omega t}, \qquad i = 1, \cdots . n, \tag{1.2}$$

where A_i is a constant amplitude, and ω is the angular frequency the eigenvalues of which may be obtained from the equation

$$\begin{vmatrix} -\omega^2 + \omega_{11}^2 , & \omega_{12}^2 , & \cdots & \omega_{1n}^2 \\ \omega_{21}^2 , & -\omega^2 + \omega_{22}^2 , & \cdots & \omega_{2n}^2 \\ \cdots & \cdots & \cdots & \cdots \\ \omega_{n1}^2 , & \omega_{n2}^2 , & \cdots & -\omega^2 + \omega_{nn}^2 \end{vmatrix} = 0, \tag{1.3}$$

where

$$\frac{k_{ij}}{Q_i} = \omega_{ij}^2 . \tag{1.4}$$

There are also other mechanical systems leading to the same type of frequency equation (1.3), e.g. the free vibrations of a rigid body, supported on elastic springs, if the coordinate system is properly chosen. In this case we have $n \leq 6$.

2. Expansion of frequency determinants. A frequency or "secular" determinant of the form

$$\Delta(x) = \begin{vmatrix} a_1 + x, & a_{12} , & \cdots & a_{1n} \\ a_{21} , & a_2 + x, & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} , & a_{n2} , & \cdots & a_n + x \end{vmatrix}$$

*Received April 23, 1951.

may be developed in the series

$$\Delta(x) = \prod_i^n (a_i + x) + \sum_{i<k}^n \begin{vmatrix} 0, & a_{ik} \\ a_{ki}, & 0 \end{vmatrix} \prod_i^{n\,(i,k)} (a_i + x)$$

$$+ \sum_{i<k<l}^n \begin{vmatrix} 0, & a_{ik}, & a_{il} \\ a_{ki}, & 0, & a_{kl} \\ a_{li}, & a_{lk}, & 0 \end{vmatrix} \prod_i^{n\,(i,k,l)} (a_i + x) + \cdots$$

$$+ \sum_{i<j}^n \begin{vmatrix} 0, & a_{12}, & \cdots & a_{1n} \\ a_{21}, & 0, & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}, & a_{n2}, & \cdots & 0 \end{vmatrix}^{(i,j)} (a_i + x)(a_j + x)$$

$$+ \sum_i^n \begin{vmatrix} 0, & a_{12}, & \cdots & a_{1n} \\ a_{21}, & 0, & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}, & a_{n2}, & \cdots & 0 \end{vmatrix}^{(i)} (a_i + x)$$

$$+ \begin{vmatrix} 0, & a_{12}, & \cdots & a_{1n} \\ a_{21}, & 0, & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}, & a_{n2}, & \cdots & 0 \end{vmatrix}, \qquad (2.1)$$

which may be obtained by induction or by considering Δ as function of n variables $y_i = a_i + x$ and using McLaurins development for this function of n variables.*

In (2.1) upper indices on a determinant indicate that the minor, obtained by suppressing the corresponding rows and columns, is to be taken; upper indices on a product sign indicate that the corresponding factors in the product are to be replaced by unity. It is of interest to note that products of the type

$$\prod_i^{n\,(i)} (a_i + x),$$

that is products of degree $n - 1$ in x, do not occur in (2.1).

*The latter method was suggested to the author by Mr. L. E. Zachrisson.

The expansion (2.1) may be used with advantage to approximate the roots of the determinantal equation (1.3).

3. The roots of the equation $\Delta(x) = 0$. Under the assumption that the diagonal terms $a_1, \cdots a_n$ of the matrix are all different and absolutely greater than the terms a_{ij}, $(i \neq j)$ outside the diagonal*, it is easy to obtain *approximate expressions for all the roots of the equation* $\Delta(x) = 0$.

To this effect we multiply all the terms a_{ij} with a common factor λ, so that the expansion (2.1) takes the form of a polynomial of n:th degree in λ, where the first degree term is missing.

Putting the r:th root x_r of the equation $\Delta(x) = 0$ in the form of a power series in λ, and determining its coefficients by introducing this expression into the equation, we obtain

$$x_r = -a_r + \lambda^2 \frac{\sum_{i<k}^{n} \begin{vmatrix} 0, & a_{jk} \\ a_{kj}, & 0 \end{vmatrix} \prod_{i}^{n\,(j,k,r)} (a_i - a_r)}{\prod_{i}^{n\,(r)} (a_i - a_r)} + \lambda^3(\cdots), \qquad (r = 1, \cdots n) \tag{3.1}$$

This power series in λ will converge for sufficiently small $|\lambda|$, which means a condition for the magnitude of the matrix elements $|a_{ij}|$.

The condition, imposed at the outset, that *all* the diagonal terms a_i be different and dominate the a_{ij} is not essential.

In fact, it is easily seen that in the simple cases $n = 2$ and 3 it will make no difference, if *one* of the diagonal terms be included among the small terms, e.g. if one of the terms a_i be multiplied with λ just as the terms a_{ij} above. In these cases similar, though somewhat modified expansions for the roots x_r will be possible. On the other hand it may be seen in the case $n = 3$, that if *two* of the diagonal terms, e.g. a_2 and a_3, be multiplied with λ, but not a_1, then an expansion of the type (3.1) will be possible for x_1 but not for x_2 and x_3, unless certain conditions be imposed upon the matrix.

We shall leave the question at this point and give an illustration of the treatment of a special case of practical importance in the next section.

4. Application to spring mounted foundations.

Spring mounted or resilient foundations for stationary machinery, prime movers etc. are built in a great variety of types and forms. In many cases, though, the design will be something like the one shown in Fig. 1, and the assumptions made below will hold, at least approximately. We shall consider the foundation as rigid and the springs to have spring constants according to Hooke's law. In this case there are in general six degrees of freedom of the system. Comparatively few references treat the problem with such generality [1]-[4], [6].†

If the coordinate system be chosen so that its axes coincide with the principal axes of inertia at the centre of gravity C.G. of the foundation, the frequency equation of the motion will have the form (1.3). We shall assume the axis C.G.-3 to be vertical—which is often true or nearly true—so that the axes C.G.-1 and C.G.-2 are to be found in a horizontal plane through C.G. Further we take q_1, q_2, q_3 to be translations of the founda-

*This assumption corresponds to what is generally termed a "weak coupling" of the system.

†Numbers in brackets refer to the bibliography at the end of the paper.

dation in directions of the coordinate axes and q_4/r_1, q_5/r_2, q_6/r_3 to be rotations with respect to the same axes, where r_1, r_2, r_3 are the corresponding radii of gyration of the foundation.

Furthermore, we assume the springs to have the following properties, often fulfilled in practice:

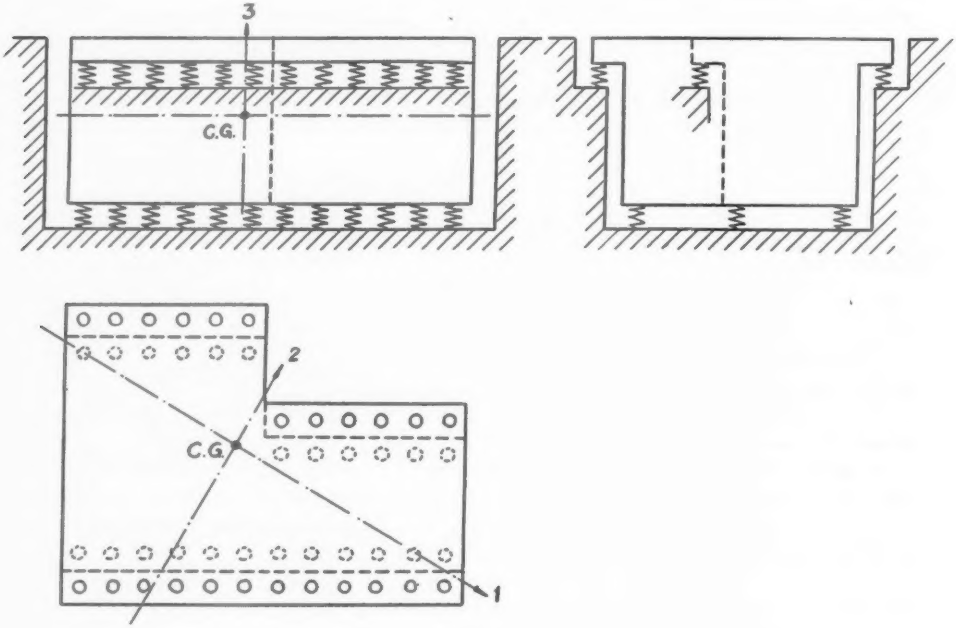

FIG. 1.

(a) all springs have vertical axes, e.g. are statically submitted to pure compression;

(b) each spring has a spring constant in vertical direction much larger than the one in a horizontal direction (for parallel displacements of the end faces of the springs). This will hold true f.i. for the ordinary type of rubber springs, used in this connection [5], [6], but it will generally not be the case for ordinary helical steel springs. Fig. 2 shows the relation between horizontal and vertical spring constant $k_h : k_v$ for different values of the relative compression ξ for the ordinary type of cylindrical rubber springs, vulcanized to steel plates, according to [6].

(c) All springs have over-all dimensions small as compared with the foundation. If $h_{1\nu}$, $h_{2\nu}$, $h_{3\nu}$ are the coordinates of the point of application of the ν:th spring and its spring constants in direction of the axes correspondingly $\kappa k_{1\nu}$, $\kappa k_{2\nu}$, $k_{3\nu}$ where, according to (b), κ may be taken as a small dimension-less quantity and $k_{1\nu}$, $k_{2\nu}$ and $k_{3\nu}$ are all of the same order of magnitude, then the springs may be arranged by the designer so as to fulfill the conditions

$$\sum_\nu k_{3\nu}h_{2\nu} = 0, \qquad \sum_\nu k_{3\nu}h_{1\nu} = 0, \qquad \sum_\nu k_{3\nu}h_{1\nu}h_{2\nu} = 0, \tag{4.1}$$

the summations being extended over all the springs. From fig. 2 it is seen that κ for rubber mountings, used in practice, will be of the order 0.1 – 0.15 (shaded area).

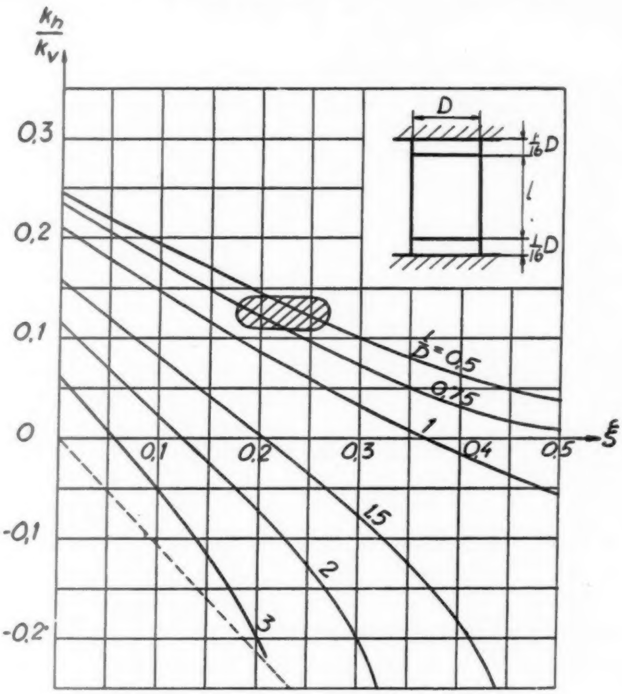

FIG. 2.

From (a), (b) and (c) it follows that the potential energy takes the simple form

$$W = \tfrac{1}{2} \sum_{\nu} (\kappa k_{1\nu}\delta_{1\nu}^2 + \kappa k_{2\nu}\delta_{2\nu}^2 + k_{3\nu}\delta_{3\nu}^2),$$

where

$$\delta_{1\nu} = q_1 + q_5 h_{3\nu}/r_2 - q_6 h_{2\nu}/r_3 ,$$

$$\delta_{2\nu} = q_2 + q_6 h_{1\nu}/r_3 - q_4 h_{3\nu}/r_1 ,$$

$$\delta_{3\nu} = q_3 + q_4 h_{2\nu}/r_1 - q_5 h_{1\nu}/r_2 .$$

The assumptions (a), (b), (c) make possible further considerable simplification of the frequency equation (1.3). Due to (c) we may write

$$\omega_{ij}^2 = \omega_{ij}^{*2} \pm \kappa \bar{\omega}_{ij}^2 ,$$

where the upper or lower sign should be taken so as to make $\bar{\omega}_{ij}$ real. Then ω_{ij}^* and $\bar{\omega}_{ij}$ are of the same order of magnitude, if different from zero at all.

Accordingly, we put (Q = total mass of foundation)

$$
\begin{aligned}
&\omega_{11}^2 = \kappa\bar{\omega}_{11}^2 = \frac{\kappa}{Q}\sum_\nu k_{1\nu}\,, \qquad \omega_{15}^2 = \kappa\bar{\omega}_{15}^2 = \frac{\kappa}{Qr_2}\sum_\nu k_{1\nu}h_{3\nu}\,,\\
&\omega_{16}^2 = -\kappa\bar{\omega}_{16}^2 = -\frac{\kappa}{Qr_3}\sum_\nu k_{1\nu}h_{2\nu}\,, \qquad \omega_{22}^2 = \kappa\bar{\omega}_{22}^2 = \frac{\kappa}{Q}\sum_\nu k_{2\nu}\,,\\
&\omega_{24}^2 = -\kappa\bar{\omega}_{24}^2 = -\frac{\kappa}{Qr_1}\sum_\nu k_{2\nu}h_{3\nu}\,, \qquad \omega_{26}^2 = \kappa\bar{\omega}_{26}^2 = \frac{\kappa}{Qr_3}\sum_\nu k_{2\nu}h_{1\nu}\,,\\
&\omega_{33}^2 = \frac{1}{Q}\sum_\nu k_{3\nu}\,, \qquad \omega_{42}^2 = -\kappa\bar{\omega}_{42}^2 = -\frac{\kappa}{Qr_1}\sum_\nu k_{2\nu}h_{3\nu}\,,\\
&\omega_{44}^2 = \omega_{44}^{*2} + \kappa\bar{\omega}_{44}^2 = \frac{1}{Qr_1^2}\sum_\nu (k_{3\nu}h_{2\nu}^2 + \kappa k_{2\nu}h_{3\nu}^2),\\
&\omega_{46}^2 = -\kappa\bar{\omega}_{46}^2 = -\frac{\kappa}{Qr_1r_3}\sum_\nu k_{2\nu}h_{1\nu}h_{3\nu}\,, \qquad \omega_{51}^2 = \kappa\bar{\omega}_{51}^2 = \frac{\kappa}{Qr_2}\sum_\nu k_{1\nu}h_{3\nu}\,,\\
&\omega_{56}^2 = -\kappa\bar{\omega}_{56}^2 = -\frac{\kappa}{Qr_2r_3}\sum_\nu k_{1\nu}h_{2\nu}h_{3\nu}\,,\\
&\omega_{55}^2 = \omega_{55}^{*2} + \kappa\bar{\omega}_{55}^2 = \frac{1}{Qr_2^2}\sum_\nu (k_{3\nu}h_{1\nu}^2 + \kappa k_{1\nu}h_{3\nu}^2),\\
&\omega_{61}^2 = -\kappa\bar{\omega}_{61}^2 = -\frac{\kappa}{Qr_3}\sum_\nu k_{1\nu}h_{2\nu}\,, \qquad \omega_{62}^2 = \kappa\bar{\omega}_{62}^2 = \frac{\kappa}{Qr_3}\sum_\nu k_{2\nu}h_{1\nu}\,,\\
&\omega_{64}^2 = -\kappa\bar{\omega}_{64}^2 = -\frac{\kappa}{Qr_1r_3}\sum_\nu k_{2\nu}h_{1\nu}h_{3\nu}\,, \qquad \omega_{65}^2 = -\kappa\bar{\omega}_{65}^2 = -\frac{\kappa}{Qr_2r_3}\sum_\nu k_{1\nu}h_{2\nu}h_{3\nu}\,,\\
&\omega_{66}^2 = \kappa\bar{\omega}_{66}^2 = \frac{\kappa}{Qr_3^2}\sum_\nu (k_{1\nu}h_{2\nu}^2 + k_{2\nu}h_{1\nu}^2).
\end{aligned} \tag{4.2}
$$

All the rest of the quantities ω_{ij}^2 vanish

$$\omega_{12}^2 = \omega_{13}^2 = \omega_{14}^2 = \omega_{21}^2 = \omega_{23}^2 = \omega_{25}^2 = \omega_{31}^2 = \omega_{32}^2 = \omega_{36}^2 = \omega_{41}^2 = \omega_{52}^2 = \omega_{63}^2 = 0. \tag{4.3}$$

for reasons of symmetry, and

$$\omega_{34}^2 = \omega_{35}^2 = \omega_{43}^2 = \omega_{45}^2 = \omega_{53}^2 = \omega_{54}^2 = 0 \tag{4.4}$$

due to the relations (4.1).

If the expressions (4.2) to (4.4) for ω_{ij}^2 are introduced into (1.3) the determinant will take the following form:

$$\Delta(-\omega^2) =$$

$$\begin{vmatrix} -\omega^2+\kappa\bar{\omega}_{11}^2, & 0, & 0, & 0, & \kappa\bar{\omega}_{15}^2, & -\kappa\bar{\omega}_{16}^2 \\ 0, & -\omega^2+\kappa\bar{\omega}_{22}^2, & 0, & -\kappa\bar{\omega}_{24}^2, & 0, & \kappa\bar{\omega}_{26}^2 \\ 0, & 0, & -\omega^2+\omega_{33}^2, & 0, & 0, & 0 \\ 0, & -\kappa\bar{\omega}_{42}^2, & 0, & -\omega^2+\omega_{44}^{*2}+\kappa\bar{\omega}_{44}^2, & 0, & -\kappa\bar{\omega}_{46}^2 \\ \kappa\bar{\omega}_{61}^2, & 0, & 0, & 0, & -\omega^2+\omega_{55}^{*2}+\kappa\bar{\omega}_{55}^2, & -\kappa\bar{\omega}_{56}^2 \\ -\kappa\bar{\omega}_{61}^2, & \kappa\bar{\omega}_{62}^2, & 0, & -\kappa\bar{\omega}_{64}^2, & -\kappa\bar{\omega}_{65}^2, & -\omega^2+\kappa\bar{\omega}_{66}^2 \end{vmatrix} \tag{4.5}$$

Due to the comparatively great number of small terms ω_{ii}^2 $(i = 1, 2, 6)$ in the diagonal, it is not possible to obtain expansions in power series in κ of the type (3.1) for the roots of the equation.

If, on the other hand, we submit the matrix to the additional conditions

$$\bar{\omega}_{16}^2 = \bar{\omega}_{61}^2 = \bar{\omega}_{26}^2 = \bar{\omega}_{62}^2 = 0, \tag{4.6}$$

which are equivalent to the conditions

$$\sum_\nu k_{1\nu}h_{2\nu} = 0, \qquad \sum_\nu k_{2\nu}h_{1\nu} = 0, \tag{4.7}$$

such developments will hold. It is to be noted that in many practical cases the conditions (4.7) will be a consequence of the two first conditions (4.1), due to the properties of the spring elements, such as disclosed by fig. 2.

We may now put

$$\begin{aligned} &\omega_1^2 = \kappa\bar{\omega}_{11}^2 + \kappa^2\mu_1 + \cdots, \qquad \omega_2^2 = \kappa\bar{\omega}_{22}^2 + \kappa^2\mu_2 + \cdots \\ &[\omega_3^2 = \omega_{33}^2], \qquad \omega_4^2 = \omega_{44}^{*2} + \kappa\bar{\omega}_{44}^2 + \kappa^2\mu_4 + \cdots \\ &\omega_5^2 = \omega_{55}^{*2} + \kappa\bar{\omega}_{55}^2 + \kappa^2\mu_5 + \cdots, \qquad \omega_6^2 = \kappa\bar{\omega}_{66}^2 + \kappa^2\mu_6 + \cdots \end{aligned} \tag{4.8}$$

Retaining the first term of the development (2.1), as applied to (4.5), untouched and ordering the rest of the terms according to ascending powers of κ^2, we may, after canceling the common factor $\omega^2 - \omega_{33}^2$, insert successively the expressions (4.8) in (2.1) and thus determine the unknown coefficients $\mu_1, \cdots, \mu_6$. Thus we finally get the simple solution

$$\begin{aligned} &\mu_1 = -\frac{\bar{\omega}_{15}^2\bar{\omega}_{51}^2}{\omega_{55}^{*2}}, \qquad \mu_2 = -\frac{\bar{\omega}_{24}^2\bar{\omega}_{42}^2}{\omega_{44}^{*2}}, \\ &\mu_4 = \frac{\bar{\omega}_{46}^2\bar{\omega}_{64}^2}{\omega_{44}^{*2}}, \qquad \mu_5 = \frac{\bar{\omega}_{56}^2\bar{\omega}_{65}^2}{\omega_{55}^{*2}}, \\ &\mu_6 = -\frac{\bar{\omega}_{46}^2\bar{\omega}_{64}^2\omega_{55}^{*2} + \bar{\omega}_{56}^2\bar{\omega}_{65}^2\omega_{44}^{*2}}{\omega_{44}^{*2}\omega_{55}^{*2}}. \end{aligned} \tag{4.9}$$

5. Conclusions

By using an expansion of the type (2.1) of the determinantal frequency equation (1.3), obtained for ordinary mechanical systems with n degrees of freedom, it is easy to obtain approximate expressions (3.1), supposing "weak coupling" and distinct eigenvalues. By examples it is shown, that similar developments of the eigenvalues as (3.1) will hold also in cases, where at least some of the terms in the principal diagonal of the matrix are of the same order as the coupling terms.

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BOOK REVIEWS

Electromagnetic problems of microwave theory. By H. Motz. Methuen & Co. Ltd., London, and John Wiley & Sons, Inc., New York, 1951. vii + 184 pp. $2.00.

This book is a new Methuen Monograph dealing in Chapter one with a summary of general topics in microwave methods including velocity modulated tubes, travelling wave tubes, resonators, cavity magnetrons, methods of detection, wavemeters, standing wave meters, the Smith chart, and matching. Chapters two and three deal with a detailed discussion of velocity modulation and Klystron theory following the work of Webster. Chapter four is entitled Mode Selection in Cavity Magnetrons and includes the Fourier Analysis of the rotating waves and conditions for synchronism of the electrons with the rotating field components. Some discussion of mode stability is given together with an estimate of magnetron efficiency.

Chapter five discusses the field relations in wave guides in orthogonal curvilinear systems. Application is made to guides of rectangular and circular cross section. The use of series expansions in terms of normal mode solutions is illustrated. The transmission line analogy is discussed briefly.

Calculation of electromagnetic fields in cavities and guides of complex shape is discussed in some detail in chapter six where relaxation and finite difference methods are used. The methods are illustrated with resonator gaps. The determination of the field pattern for higher modes is also discussed briefly. The analytical treatment of corrugated wave guides, as done by Wilkinshaw, is outlined very well in the space of about six pages.

Chapter seven is concerned with the impedance of an antenna in a wave guide. The problem of coupling into wave guides by means of straight wires or loops, as treated by the Toronto group, is presented in a very satisfactory way. The theory of discontinuities in wave guides as worked out by Bethe and Schwinger is outlined and applied to transverse windows, changes in cross section, and bends. The methods are applicable only to thin windows and sudden changes in cross section but a numerical method, applicable to any type of discontinuity, is outlined.

A large amount of useful material is packed into the 180 pages that make up this monograph. The writer did a very good job of discussing the problems which he includes. A person actively working in microwave theory will probably find this book useful as well as the person looking for general information.

ROHN TRUELL

Ordinary non-linear differential equations in engineering and physical sciences. By N. W. McLachlan. Oxford at the Clarendon Press, 1950. vi + 201 pp. $4.25.

The contents of this book are of essentially two categories. Following a general introduction, the second chapter deals with examples of non-linear equations which can be integrated explicitly. The third chapter is concerned with the particular situation where the explicit integration requires the use of elliptic functions. The remainder of the book deals with the second category, i.e., the non-linear equations which arise in vibration problems.

In chapter four, the Van der Pol equation with small parameter, and the non-linear restoring force problem, are treated in detail. Emphasis is given to the physical phenomena involved as well as to the mathematical solutions. The relaxation oscillations of the Van der Pol equation are discussed very briefly. The method of slowly varying amplitude and phase and the equivalent linear equation are discussed in chapters five and six. Chapter seven is devoted to equations with periodic coefficients and chapter eight to numerical and graphical methods of solutions. Almost every equation in the book is illustrated by a physical problem so that the book should be of considerable interest to engineers.

G. F. CARRIER

Integraltafel. Zweiter Teil: Bestimmte Integrale. By Wolfgang Gröbner and Nikolaus Hofreiter. Springer-Verlag, Wien and Innsbruck, 1950. vi + 551 pp. $5.80.

This very complete table of definite integrals is a companion volume to Integraltafel (Unbestimmte Integrale) of indefinite integrals by the same authors, published in 1949 (reviewed QAM, January 1950). This book contains approximately two thousand definite integrals including some which are not found in Bierens de Haan. The symbols are large and clear and the general make-up of the table is very satisfactory. The only change that this reviewer would ask for is specification of the limits of integration in the table of contents. The table of contents is as follows: Symbols and Notation, Methods of Calculating Definite Integrals, General Integral Formulae. (1) Rational Integrands (detailed listing omitted). Orthogonal Polynomials, including Legendre, Hypergeometric, Tschebyscheff, Associated Legendre, Laguerre and Hermite Polynomials (detailed listing omitted). (2) Algebraic Irrational Integrands, including elliptic integrals of Legendre and Weierstrass Canonical forms (detailed listing omitted). (3) Elementary Transcendental Integrands. Integrands of the form $R(\exp \lambda x, \exp \mu x, \cdots)$, $[\exp(-sx)]f(x)$, $R(x, \exp \lambda x)$, $R(x, \exp f(x))$, $f(\log x)$, $\log [g(x)]$, Euler Dilogarithm, $f(x) (\log x)^n$ with $f(x)$ rational, irrational and transcendental, $f(x) \log [g(x)]$, $F[x, \log f(x)]$, (exponential, logarithmic, sine and cosine integrals), $f(\sin x, \cos x)$, $f(\sin ax, \cos bx, \cdots)$, $F[x, \sin ax, \cos bx]$, $F[x, \sin f(x), \cos g(x), \cdots]$, $F[\exp ax, \sin bx, \cos cx]$, $F[x, \exp ax, \sin bx, \cos cx]$, $F[x, \exp f(x), \sin g(x), \cos h(x)]$, $F[x, \log f(x), \sin g(x), \cos h(x)]$, $F[x, \text{arc} \sin x, \text{arc} \cos x]$, $F[x, \text{arc} \tan x, \text{arc cotan}\, x]$, $R[\exp \lambda x, \sinh ax, \cosh bx]$, $R[x, \sinh ax, \cosh bx]$, $F[f(x), \sinh ax, \cosh bx]$. Integrals of $\sinh^{-1} x$, $\cosh^{-1} x$, $\tanh^{-1} x$, $\text{cotanh}^{-1} x$. (4) Euler Integrals. Gamma function, Beta function, products of powers of linear expressions with general exponents, products of powers of binomial expressions with general exponents, products of powers of higher order expressions with general exponents. (5) Integrals of Cylinder functions. Bessel functions, Bessel functions with imaginary arguments, Integrals of the form $F[x, Z_n(x)]$, $F[x, \exp x, \log x, Z_n(x)]$, $F[x, \sin x, \cos x, Z_n(x)]$, $F[x, Z_n(x), Z_\mu(x)]$.

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